Rank 2 stable sheaves on toric threefolds: classical and virtual counts

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Assuming $\gcd(r, c_1 \cdot H^{n-1}) = 1$, the moduli space is projective.

Generating function of Euler characteristics

Consider generating function of Euler characteristics

$$G_{c_1,\ldots,c_{n-1}}(q) = \sum_{c_n} e(\mathcal{M}_X^H(r, c_1, \ldots, c_n))q^{c_n}.$$

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Reflexive hulls

For torsion free sheaf $\mathcal{E}$ define $\mathcal{E}^* = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$. Then $\mathcal{E} \hookrightarrow \mathcal{E}^{**}$ and $\mathcal{E}^{**}$ is called reflexive hull of $\mathcal{E}$.

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Moduli of reflexive sheaves (non-compact!): $\mathcal{N}_X^H(r, c_1, c_2, c_3)$.

There exists a constructible map

$$()^{**} : \mathcal{M}_X^H(r, c_1, c_2, c_3) \to \coprod_{c_2', c_3'} \mathcal{N}_X^H(r, c_1, c_2', c_3').$$

fibre over R is

$$Quot(R, c_2'', c_3'') := \{R \to Q \to 0 \mid c_2(Q) = c_2'', \;\; c_3(Q) = c_3''\}.$$

Reflexive hulls

Idea: When X toric with torus T compute $e(\mathcal{M}_X^H(r, c_1, c_2, c_3))$ from:

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G-Kool (2013) Rank 2

For X nonsingular toric threefold $\mathcal{N}_X^H(2, c_1, c_2', c_3')^T$ can be described explicitly. It is a union of configuration spaces of distinct points on $\mathbb{P}^1$.

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For any c_1, c_2, there are explicit subsets $D_i(c_1, c_2) \subset \mathbb{Z}^4_{\geq 0}$, $i = 1, 2, 3$ defined by explicit polynomial equalities and inequalities, such that

$$G^{refl}_{2,c_1,c_2}(q) = \sum_{\mathbf{v} \in D_1(c_1,c_2)} -q^{C_1(\mathbf{v})} + \sum_{\mathbf{v} \in D_2(c_1,c_2)} 6q^{C_2(\mathbf{v})} + \sum_{\mathbf{v} \in D_3(c_1,c_2)} 4q^{C_3(\mathbf{v})}$$

$$C_1(\mathbf{v}) = \sum_{1 \leq i < j < k \leq 4} v_i v_j v_k, \; C_2(\mathbf{v}) = (v_1 + v_2) v_3 v_4, \; C_3(\mathbf{v}) = v_1 v_2 v_3.$$

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E.g. For $c_1 = -1$ and $c_2 = 1, 2, 3, \ldots$

$$G^{refl}_{2,-1,c_2}(q) = 4q, 24q^4, -4q^7 + 36q^9, \ldots$$

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Hartshorne 1980, G-Kool (only for T-equivariant)

R rank 2 stable reflexive sheaf on $X = \mathbb{P}^3$ with Chern classes c_1, c_2, c_3.

1. $c_3 = c_1 c_2 \mod 2$, if $c_1 \in \{-1, 0\}$, then $c_2 > 0$,
2. if $c_1 = -1$, then $0 \leq c_3 \leq c_2^2$, and if $c_1 = 0$, then $0 \leq c_3 \leq c_2^2 - c_2 + 2$. Both upper bounds are sharp.

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G-Kool

For $c_2 > 1, \quad e(\mathcal{N}_{\mathbb{P}^3}(2, -1, c_2, c_2^2)) = 12 c_2$.

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Consequences:

- If $\mathcal{E} \in \mathcal{M}_X^H(2, c_1, c_2, c_3)$ then the quotients $\mathcal{E}^{**}/\mathcal{E}$ are 0-dimensional.
- $\mathcal{M}_X^H(2, c_1, c_2, c_3)^T$ is a finite disjoint union of $Quot(R, s - c_3)^T$ where R is reflexive and s is the length of singularity of R (i.e. $c_2'' = 0, c_3'' = s - c_3$).
- We have universal families.

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G-Kool-Young

If c_1, c_2 satisfy the assumption, then $G_{2,c_1,c_2}(q)$ is given by $M(q)^{2e(X)}$ times

$$\sum_{R \text{ locally free}} 1 + \sum_{R \text{ singular}} \prod_{i=1}^{v_1(R)} \prod_{j=1}^{v_2(R)} \prod_{k=1}^{v_3(R)} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}.$$

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Corollary, $X = \mathbb{P}^3$, Rank 2

$$G_{-1,1}(q) = 4(q + q^{-1})M(q^{-2})^8.$$

$$G_{-1,2}(q) = 12\left(\frac{2q^{-4} - q^{-2} + 1 - 4q^2 + 3q^4 + 5q^8}{(1-q^2)^2}\right)M(q^{-2})^8.$$

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For $c_2 = 2$ the quotients are no longer 0-dimensional. For $c_2 = 3$ the T-fixed reflexive hulls are no longer isolated.

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For singular schemes $\mathcal{M}$, the cotangent bundle is replaced by the cotangent complex $L^\bullet_{\mathcal{M}}$ with $h^0(L^\bullet_{\mathcal{M}}) = \Omega_{\mathcal{M}}$ and $h^i(L^\bullet_{\mathcal{M}}) = 0$ for $i > 0$.
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A perfect obstruction theory on $\mathcal{M}$ consists of a 2-term complex $E^\bullet = [E^{-1} \to E^0]$ of vector bundles on $\mathcal{M}$ together with a morphism (in the derived category) $\phi : E^\bullet \to L^\bullet \mathcal{M}$ such that

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Under the assumption, we have $\beta = 0$, and the obstruction theory is symmetric for any X.

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CY3 specialization

Conjecture (G-Kool-Young)

$W_{R_\alpha,\emptyset,\emptyset,\emptyset}(q)|_{s_1+s_2+s_3=0}$ is equal to $M(q)^2$ times

$$\begin{cases} 1 & R \text{ locally free} \\ \prod_{i=1}^{v_1(R)} \prod_{j=1}^{v_2(R)} \prod_{k=1}^{v_3(R)} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} & R \text{ singular} \end{cases}$$

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One is about the smoothness of the T_0-fixed locus, and the other is about the parity of the constant terms after the specialization $t_1 t_2 t_3 = 1$.

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Equivalent data: collection of vector spaces $\{F(k_1, k_2, k_3)\}_{(k_1,k_2,k_3)\in\mathbb{Z}^3}$ and linear maps

$$\chi_1(k_1, k_2, k_3) : F(k_1, k_2, k_3) \to F(k_1 + 1, k_2, k_3),$$

$$\chi_2(k_1, k_2, k_3) : F(k_1, k_2, k_3) \to F(k_1, k_2 + 1, k_3),$$

$$\chi_3(k_1, k_2, k_3) : F(k_1, k_2, k_3) \to F(k_1, k_2, k_3 + 1),$$

such that $\chi_i \circ \chi_j = \chi_j \circ \chi_i$ for all $i, j, (k_1, k_2, k_3)$.

Localization

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$\mathcal{F}$ reflexive $\Leftrightarrow$ $\exists$ filtrations

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s.t. $\quad F(k_1, k_2, k_3) = F(k_1, \infty, \infty) \cap F(\infty, k_2, \infty) \cap F(\infty, \infty, k_3).$

rank 2 reflexive sheaves

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1. three integers $u_i \in \mathbb{Z}$ where flag i jumps from 0 to $p_i \in \mathbb{P}^1$,
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Classification of stable reflexive sheaves (G-Kool)

1. Type I: $0 < v_i < v_j + v_k + v_l \ \forall \{i,j,k,l\} = \{1,2,3,4\}$ and all p_i are mutually distinct,
2. Type II: $v_1, v_2, v_3, v_4 > 0$, $\exists \{i,j,k,l\} = \{1,2,3,4\}$ such that $v_i + v_j < v_k + v_l, v_k < v_i + v_j + v_l, v_l < v_i + v_j + v_k$, $p_i = p_j$, and p_j, p_k, p_l are mutually distinct,
3. Type III: $\exists \{i,j,k,l\} = \{1,2,3,4\}$ such that $v_i = 0, v_j, v_k, v_l > 0, v_j < v_k + v_l, v_k < v_j + v_l, v_l < v_j + v_k$, and p_j, p_k, p_l are mutually distinct.

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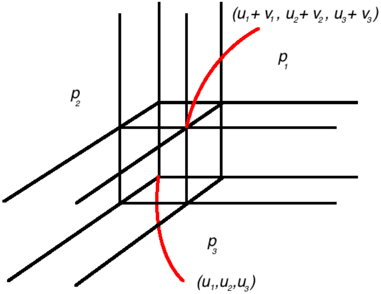

$O := (u_1, u_2, u_3)$, $S := (u_1 + v_1, u_2 + v_2, u_3 + v_3)$,
$P_1 := (u_1, u_2 + v_2, u_3 + v_3)$, $P_2 := (u_1 + v_1, u_2, u_3 + v_3)$,
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$\mathrm{Quot}(R, n)^T$

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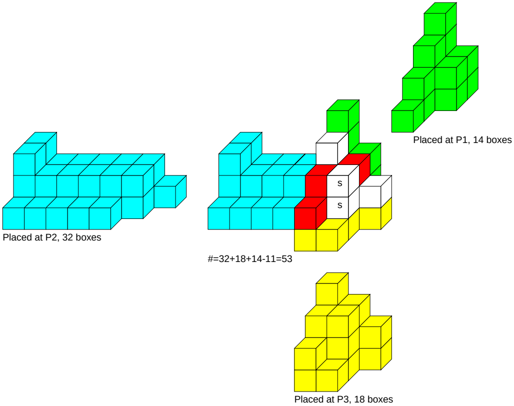

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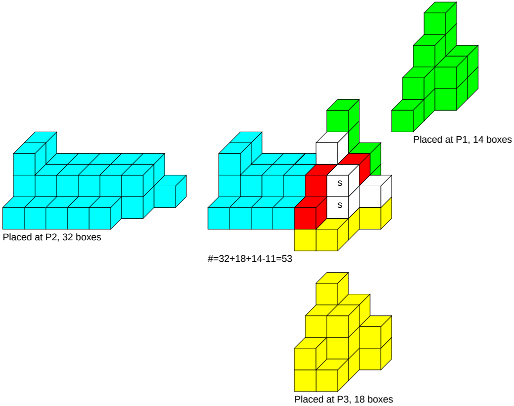

Three components of white boxes: Two are unsupported (hence unlabeled), and one is supported (labeled with $s \in \mathbb{P}^1$).

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Define

$$G_{\mathbf{u},\mathbf{v}}(q) = \sum_{[\boldsymbol{\pi}]} 2^{k(\boldsymbol{\pi})} q^{\#(\boldsymbol{\pi})}$$

sum over equivalence classes of triple partitions.

Formula

(G-Kool-Young)

For any integers u_1, u_2, u_3 and $v_1, v_2, v_3 > 0$ we have

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In the combinatorial proof the role of B is not clear, but B plays a big role in geometric proof we found later.

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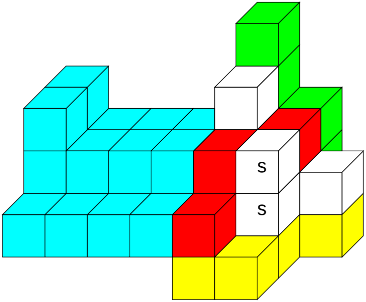

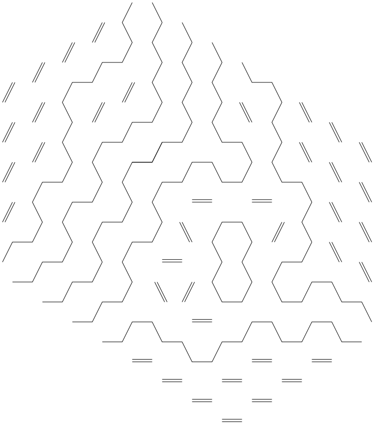

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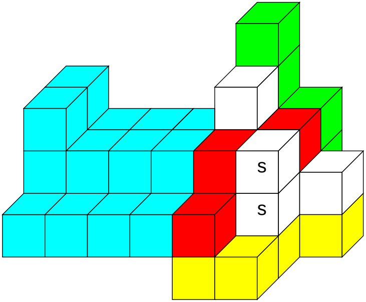

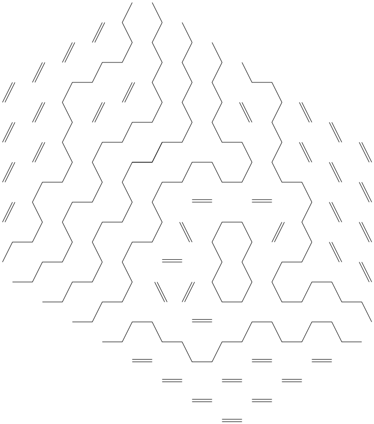

Labeled boxes correspond to loops in the dimer model. There is no bijection between the triple of partitions and double dimer models. However, their generating functions match.

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$$\begin{aligned} R \in \operatorname{Ext}^1(I_C, L) &\cong \operatorname{Ext}^2(\mathcal{O}_C, L) \\ &\cong \operatorname{Ext}^1(L, \mathcal{O}_C \otimes \omega_X)^* \\ &\cong H^1(C, \omega_X \otimes L^{-1}|_C)^* \\ &\cong H^0(C, \omega_C \otimes \omega_X^{-1} \otimes L). \end{aligned}$$

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Applying $\mathcal{H}om(\cdot, L)$ to s.e.s above gives

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Note $\omega_C = \mathcal{E}xt^2(\mathcal{O}_C, \omega_X)$ so $\omega_C \otimes \omega_X^{-1} \otimes L = (\mathcal{O}_C)_L^D$.

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Note $\omega_C = \mathcal{E}xt^2(\mathcal{O}_C, \omega_X)$ so $\omega_C \otimes \omega_X^{-1} \otimes L = (\mathcal{O}_C)_L^D$.

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Ext groups

If I_C (resp. R) is the ideal sheaf of a CM curve (resp. rank 2 reflexive sheaf) and Q a 0-dimensional sheaf, the only nonzero Ext groups are

$$\mathrm{Hom}(I_C, Q), \mathrm{Ext}^1(I_C, Q) \quad \mathrm{Hom}(R, Q), Ext^1(R, Q)$$

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Furthermore,

$$\dim \mathrm{Ext}^1(I_C, Q[-1]) - \dim \mathrm{Ext}^1(Q[-1], I_C) = \ell(Q),$$
$$\dim \mathrm{Ext}^1(R, Q[-1]) - \dim \mathrm{Ext}^1(Q[-1], R) = 2\ell(Q),$$

only depend on $\ell(Q) := \mathrm{length}(Q)$.

Quot schemes

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where $\mathcal{T}$ denotes the stack of all 0-dimensional sheaves on X and "onto" refers to the subset of surjective maps in

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<u>Conclusion</u> the first nonzero Ext group $\mathrm{Ext}^1(R, Q[-1])$ governs the quot scheme $\mathrm{Quot}(R)$.

PT stable pairs

$$F \in \operatorname{Ext}^1(Q[-1], I_C) \cong \operatorname{Ext}^2(Q, I_C) \cong \operatorname{Ext}^1(Q, \mathcal{O}_C)$$

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Given $(R, \sigma) \leftrightarrow ((\mathcal{O}_C)^D_L, \xi)$ as in Serre correspondence, $\exists$ natural injection

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Conclusion the second nonzero Ext group $\mathrm{Ext}^1(Q[-1], R)$ governs the 'specific' sub-locus of PT stable pairs with support C denoted by $P(C)$.

Specific PT pairs and 1st main result

Define $P(R, \sigma) := \bigsqcup_{Q \in \mathcal{T}} \mathrm{Ext}^2(Q, R)^{pure} \subset P(C)$.

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Hall Algebra

Let $H(\mathcal{T}) := K(\mathrm{St}/\mathcal{T})$ the Grothendieck group of stacks (locally of finite type and with affine geometric stabilizers) over $\mathcal{T}$.

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For any two ($\mathcal{T}$-isomorphism classes of) $\mathcal{T}$-stacks $[U \to \mathcal{T}]$ and $[V \to \mathcal{T}]$, the product $[U * V \to \mathcal{T}]$ is defined by the Cartesian diagram

$$\begin{array}{ccccc} U * V & \longrightarrow & \mathcal{T}^2 & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow & & \\ U \times V & \longrightarrow & \mathcal{T} \times \mathcal{T}. & & \end{array}$$

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1_0 is the identity (the stack consisting of the zero sheaf with the inclusion into $\mathcal{T}$).

Some elements of Hall Algebra

Given $(R, \sigma) \leftrightarrow ((\mathcal{O}_C)_L^D, \xi)$ as in Serre correspondence, we define

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Given $(R, \sigma) \leftrightarrow ((\mathcal{O}_C)_L^D, \xi)$ as in Serre correspondence, we define

- $1_{\mathcal{T}}$ is the identity map $\mathcal{T} \to \mathcal{T}$,
- $\mathrm{Hom}(R, \cdot)$ is the stack whose fibre over $Q \in \mathcal{T}$ is $\mathrm{Hom}(R, Q)$,
- $\mathrm{Hom}(R, \cdot)^{onto}$ is the stack whose fibre over $Q \in \mathcal{T}$ is $\mathrm{Hom}(R, Q)^{onto}$,
- $\mathrm{Ext}^2(\cdot, R)$ is the stack whose fibre over $Q \in \mathcal{T}$ is $\mathrm{Ext}^2(Q, R)$,
- $\mathrm{Ext}^2(\cdot, R)^{pure}$ is the stack whose fibre over $Q \in \mathcal{T}$ is $\mathrm{Ext}^2(Q, R)^{pure}$.
- $\mathbb{C}^{r\ell(\cdot)}$ is the stack whose fibre over $Q \in \mathcal{T}$ is $\mathbb{C}^{r\ell(Q)}$.

Using the inclusion-exclusion principle, we can write $\mathrm{Hom}(R, Q)^{onto}$ as

$$\mathrm{Hom}(R, Q) - \bigsqcup_{Q_1 < Q} \mathrm{Hom}(R, Q_1) + \bigsqcup_{Q_1 < Q_2 < Q} \mathrm{Hom}(R, Q_1) - \cdots ,$$

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where $<$ denotes *strict* inclusion. Write $1_{\mathcal{T}} = 1_0 + 1_{\mathcal{T}'}$. Then $1_{\mathcal{T}}^{-1} = 1_0 - 1_{\mathcal{T}'} + 1_{\mathcal{T}'} * 1_{\mathcal{T}'} - 1_{\mathcal{T}'} * 1_{\mathcal{T}'} * 1_{\mathcal{T}'} * + \ldots$.

Some identities in Hall Algebra

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This leads to

$$\mathrm{Hom}(R,\cdot)^{onto} = \mathrm{Hom}(R,\cdot) * 1_{\mathcal{T}}^{-1}.$$

Similarly,

$$\mathrm{Ext}^2(\cdot,R)^{pure} = 1_{\mathcal{T}}^{-1} * \mathrm{Ext}^2(\cdot,R).$$

Virtual Poincaré polynomial

Now let

$$P_z(\cdot) : H(\mathcal{T}) \longrightarrow \mathbb{Q}(z)[\![q]\!],$$

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denote the virtual Poincaré polynomial. Here z is the formal variable of P_z and q keeps track of an additional grading as follows. Any element $[U \to \mathcal{T}] \in H(\mathcal{T})$ is locally of finite type so can have infinitely many components. Let $\mathcal{T}_n \subset \mathcal{T}$ be the substack of 0-dimensional sheaves of length n and define

$$P_z(U) := \sum_{n=0}^{\infty} P_z(U \times_{\mathcal{T}} \mathcal{T}_n)\, q^n.$$

Wall-crossing formula

By Serre duality and Rieman-Roch $P_z(\cdot)$ is a *Lie algebra homomorphism* to the abelian Lie algebra $\mathbb{Q}(z)[\![q]\!]$ (Joyce, Stoppa-Thomas):

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Furthermore, if both $\lim_{z\to 1} P_z(U)$ and $\lim_{z\to 1} P_z(V)$ exist then

$$\lim_{z\to 1} P_z(U * V) = \lim_{z\to 1} P_z(U) \lim_{z\to 1} P_z(V).$$

Application to our setting

Define $U := \mathrm{Hom}(R, \cdot) * (\mathbb{C}^{2\ell(\cdot)})^{-1}$ and $V := \mathbb{C}^{2\ell(\cdot)} * 1_{\mathcal{T}}^{-1}$.

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$$\lim_{z \to 1} P_z(V)(q) = \lim_{z \to 1} P_z\big(\mathrm{Hom}(\mathcal{O}_X^{\oplus 2}, \cdot)^{onto}\big)(q) = M(q)^{2e(X)},$$

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2nd main result

Theorem G-Kool

Let R be a rank 2 reflexive sheaf on a smooth projective 3-fold X. Suppose there exists a cosection $R \to \mathcal{O}_X$ cutting out a 1-dimensional closed subscheme. Then

$$\sum_{n=0}^{\infty} e\big(\operatorname{Quot}(R, n)\big) q^n = M(q)^{2e(X)} \sum_{n=0}^{\infty} e\big(\operatorname{Quot}(\mathcal{E}xt^1(R, \mathcal{O}_X), n)\big) q^n.$$

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Corollary

Let R be a singular rank 2 T-equivariant reflexive sheaf on $\mathbb{C}^3$ with homogeneous generators of weights

$(u_1+v_1, u_2+v_2, u_3), \ (u_1+v_1, u_2, u_3+v_3), \ (u_1, u_2+v_2, u_3+v_3)$.

Then

$$\sum_{n=0}^{\infty} e\big(\mathrm{Quot}(R,n)\big)\, q^n = M(q)^2 \prod_{i=1}^{v_1} \prod_{j=1}^{v_2} \prod_{k=1}^{v_3} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}.$$

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$(u_1+v_1, u_2+v_2, u_3), \ (u_1+v_1, u_2, u_3+v_3), \ (u_1, u_2+v_2, u_3+v_3)$.

Then

$$\sum_{n=0}^{\infty} e\big(\mathrm{Quot}(R,n)\big)\, q^n = M(q)^2 \prod_{i=1}^{v_1} \prod_{j=1}^{v_2} \prod_{k=1}^{v_3} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}.$$

3D partition with legs

Fix the outgoing 2D partitions $\lambda_1, \lambda_2, \lambda_3$.

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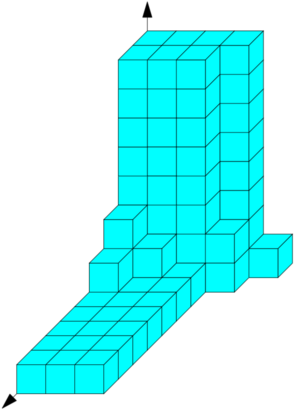

- $\lambda_1 = 1^3$
- $\lambda_2 = 2^3 1$
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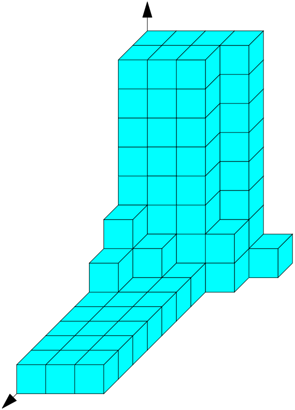

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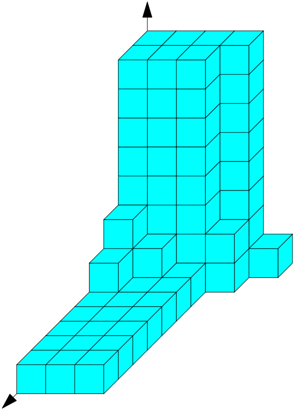

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$\sum_\pi q^{|\pi|}$ can be expressed in terms of $M(q)$ and the skewed Schur functions. (Okounkov-Reshetikhin-Vafa)

Infinite legs

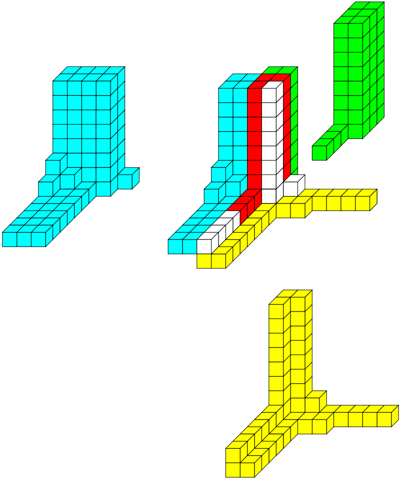
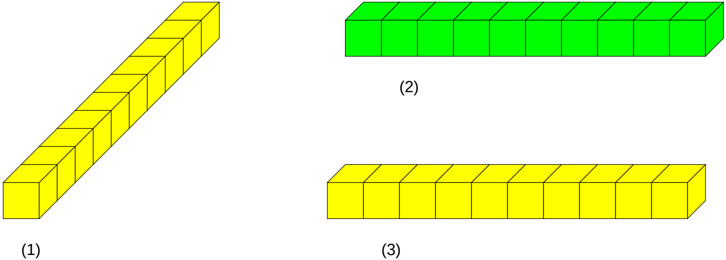


Figure: All 3D partitions are allowed to have infinite legs. Two of the white components is labelled so $k = 2$.

Example: $(v_1, v_2, v_3) = (2, 2, 1)$



$$(1) \quad M(q)^2 \frac{1 + q + q^2 + q^3 + q^4 + q^6}{1 - q}.$$

$$(2) \quad M(q)^2 \frac{1 + q + q^2 + q^3 + q^4 + q^5}{1 - q}.$$

$$(3) \quad M(q)^2 \frac{1 + q^2 + q^3 + q^4 + q^5 + q^6}{1 - q}.$$