1. (5 points) Determine whether the following series converges or diverges. Explain what test(s) you use and how each test applies.

$$\sum_{n=2}^{\infty} \frac{2^n n!}{(2n)!}.$$

**Solution:** Apply ratio test on this series.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1} (n+1)!}{(2n+2)!}}{\frac{2^n n!}{(2n)!}} = \frac{2(n+1)}{(2n+2)(2n+1)}$$

Take the limit,

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2(n+1)}{(2n+2)(2n+1)} = 0$$

So, the ratio is 0, which is less than 1. By ratio test, the series converges.

2. (5 points) Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^n x^n}{n!}.$$

**Solution:** Use generalized ratio test on this series.

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{2^{n+1} |x|^{n+1}}{(n+1)!}}{\frac{2^n |x|^n}{n!}} \right| = \frac{2}{n+1} |x|$$

When $n \to \infty$, the ratio tends to be 0. So, the radius of convergence is $R = \infty$.

3. (5 points) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}}.$$

**Solution:** Rewrite the series like below.

\[
\sum_{n=1}^{\infty} \frac{2^n}{3^n+1} = \sum_{n=1}^{\infty} \frac{1}{3} \left( \frac{2}{3} \right)^n
\]

Then, we can apply formula for geometric series (as \(|r| = 2/3 < 1\))

\[
\sum_{n=1}^{\infty} \frac{2^n}{3^n+1} = \frac{1}{3} \cdot \frac{\left( \frac{2}{3} \right)^1}{1 - \frac{2}{3}} = \frac{2}{3}.
\]

4. (5 points) Find the interval of convergence of the series

\[
\sum_{n=1}^{\infty} 2^n x^n.
\]

**Solution:** Use generalized ratio test on this series. (Generalized root test is also suitable.)

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1} |x|^{n+1}}{2^n |x|^n} = 2|x|
\]

The series will converge if \(2|x| < 1\), and will diverge if \(2|x| > 1\). So, the radius of convergence is \(R = 1/2\).

Specially, when \(x = 1/2\),

\[
\sum_{n=1}^{\infty} 2^n x^n = \sum_{n=1}^{\infty} 2^n \left( \frac{1}{2} \right)^n = \sum_{n=1}^{\infty} 1 = \infty
\]

So, \(1/2\) is excluded in the interval of convergence.

Also, when \(x = -1/2\),

\[
\sum_{n=1}^{\infty} 2^n x^n = \sum_{n=1}^{\infty} 2^n \left( -\frac{1}{2} \right)^n = \sum_{n=1}^{\infty} (-1)^n
\]

This series diverge by alternating series test. \((\lim_{n \to \infty} a_n = 1)\) So, \(-1/2\) is also excluded in the interval of convergence.

To sum up, the interval of convergence should be \((-1/2, 1/2)\).