1. (5 points) Determine whether the following series converges or diverges. Explain what test(s) you use and how each test applies.

\[ \sum_{n=1}^{\infty} \frac{(2n)!}{2^n n!}. \]

**Solution:** Apply ratio test on this series.

\[
\frac{a_{n+1}}{a_n} = \frac{\frac{(2n+2)!}{(2n)!}}{\frac{2^{n+1} (n+1)!}{2^n n!}} = \frac{(2n + 2)(2n + 1)}{2(n + 1)}
\]

Take the limit,

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(2n + 2)(2n + 1)}{2(n + 1)} = \infty
\]

So, the ratio tends to be infinity, which is greater than 1. By ratio test, the series diverges.

2. (5 points) Find the radius of convergence of the series

\[ \sum_{n=1}^{\infty} \frac{3^n}{(2n)!} x^n. \]

**Solution:** Use generalized ratio test on this series.

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{x}{3} \left( \frac{2n+2}{2n+1} \right)^n}{\frac{1}{x}} = \frac{3}{(2n + 2)(2n + 1)} |x|
\]

When \( n \to \infty \), the ratio tends to be 0. So, the radius of convergence is \( R = \infty \).

3. (5 points) Find the sum of the series

\[ \sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n}. \]
Solution: Rewrite the series like below.

\[ \sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \sum_{n=1}^{\infty} 2 \left( \frac{2}{3} \right)^n \]

Then, we can apply formula for geometric series (as \( |r| = \frac{2}{3} < 1 \))

\[ \sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = 2 \cdot \left( \frac{\frac{2}{3}}{1 - \frac{2}{3}} \right) = 4. \]

4. (5 points) Find the interval of convergence of the series

\[ \sum_{n=1}^{\infty} 2^n x^n. \]

Solution: Use generalized ratio test on this series.

\[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{3^{n+1}|x|^{n+1}}{3^n |x|^n} = 3|x| \]

The series will converge if \( 3|x| < 1 \), and will diverge if \( 3|x| > 1 \). So, the radius of convergence is \( R = \frac{1}{3} \).

Specially, when \( x = 1/3 \),

\[ \sum_{n=1}^{\infty} 3^n x^n = \sum_{n=1}^{\infty} 3^n \left( \frac{1}{3} \right)^n = \sum_{n=1}^{\infty} 1 = \infty \]

So, \( 1/3 \) is excluded in the interval of convergence.

Also, when \( x = -1/3 \),

\[ \sum_{n=1}^{\infty} 3^n x^n = \sum_{n=1}^{\infty} 3^n \left( -\frac{1}{3} \right)^n = \sum_{n=1}^{\infty} (-1)^n \]

This series diverge by alternating series test. (\( \lim_{n \to \infty} a_n = 1 \)) So, \( -1/3 \) is also excluded in the interval of convergence.

To sum up, the interval of convergence should be \((-1/3, 1/3)\).