MATH 141 - Quiz 6, Solution

2 pm

1. (5 points) Determine whether the following series converges or diverges. Explain what test(s) you use and how each test applies.

$$\sum_{n=2}^{\infty} \frac{5^{n+1} n!}{(2n+1)!}.$$

Solution: Apply ratio test on this series.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{5^{n+2}}{5^{n+1}} \frac{(n+1)!}{n!}}{\frac{(2n+3)!}{(2n+1)!}} = \frac{5(n+1)}{(2n+3)(2n+2)}$$

Take the limit,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5(n+1)}{(2n+3)(2n+2)} = 0$$

So, the ratio is 0, which is less than 1. By ratio test, the series converges.

2. (5 points) Find the radius of convergence of the series

$$\sum_{n=3}^{\infty} x^{3n}.$$

Solution: Use generalized ratio test on this series. (Generalized root test is also suitable.)

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|^{3n+3}}{|x|^{3n}} = |x|^3$$

The series will converge if $|x|^3 < 1$, and will diverge if $|x|^3 > 1$. So, the radius of convergence is $R = 1$.

3. (5 points) Find the sum of the series

$$\sum_{n=2}^{\infty} \frac{2}{(n+1)n!}.$$

Solution: We know the Taylor expansion for e^x around $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

And the radius of convergence $R = \infty$.

So, we can take $x = 1$, and have the following equality.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Now, back to solve the sum of the given series.

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{2}{(n+1)n!} &= 2\sum_{n=2}^{\infty} \frac{1}{(n+1)!} = 2\sum_{n=3}^{\infty} \frac{1}{n!} \\ &= 2\left(\sum_{n=0}^{\infty} \frac{1}{n!} - \frac{1}{0!} - \frac{1}{1!} - \frac{1}{2!}\right) = 2(e - 1 - 1 - \frac{1}{2}) = 2e - 5. \end{aligned}$$

4. (5 points) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^{n+1}x^n}{n!}.$$

Solution: Use generalized ratio test on this series.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{2^{n+2}}{2^{n+1}} \frac{|x|^{n+1}}{|x|^n}}{\frac{(n+1)!}{n!}} = \frac{2|x|}{n+1}$$

When $n \to \infty$, the ratio tends to be 0 for any x. So, the radius of convergence is $R = \infty$, and the interval of convergence is $(-\infty, \infty)$.