# MATH 141 - Quiz 6, Solution

3 pm

1. (5 points) Determine whether the following series converges or diverges. Explain what test(s) you use and how each test applies.

$$\sum_{n=2}^{\infty} \frac{(2n+1)!}{n^n}.$$

**Solution:** Apply ratio test on this series.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2n+3)!}{(2n+1)!}}{\frac{(n+1)^{(n+1)}}{n^n}} = \frac{(2n+3)(2n+2)}{\left(\frac{n+1}{n}\right)^n (n+1)}$$

Take the limit,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+3)(2n+2)}{e(n+1)} = \infty$$

Notice that, we have used the following identity.

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

So, the ratio is infinity, which is greater than 1. By ratio test, the series diverges.

2. (5 points) Find the radius of convergence of the series

$$\sum_{n=2}^{\infty} x^{4n}.$$

**Solution:** Use generalized ratio test on this series. (Generalized root test is also suitable.)

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{4n+4}}{|x|^{4n}} = |x|^4$$

The series will converge if $|x|^4 < 1$, and will diverge if $|x|^4 > 1$. So, the radius of convergence is $R = 1$.

3. (5 points) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{3}{n(n-1)!}.$$

**Solution:** We know the Taylor expansion for $e^x$ around $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

And the radius of convergence $R = \infty$.

So, we can take $x = 1$, and have the following equality.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Now, back to solve the sum of the given series.

$$\sum_{n=1}^{\infty} \frac{3}{n(n-1)!} = 3\sum_{n=1}^{\infty} \frac{1}{n!} = 3\left(\sum_{n=0}^{\infty} \frac{1}{n!} - \frac{1}{0!}\right) = 3(e-1) = 3e - 3.$$

4. (5 points) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}.$$

**Solution:** Use generalized ratio test on this series.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{|x|^{n+1}}{|x|^n}}{\frac{(n+1)(n+2)}{n(n+1)}} = \frac{n|x|}{n+2}$$

When $n \to \infty$, the ratio tends to be $|x|$. If $|x| < 1$, the series converges. If $|x| > 1$, the series diverges. So, the radius of convergence is $R = 1$.

Specially, when $x = 1$,

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

By p-test, we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. So, using comparasion rule, we have $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ also converges. (Limit comparasion rule is also suitable here.)

So, 1 is included in the interval of convergence.

Also, when $x = -1$,

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$$

From the result above, we know that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$ converges absolutely. (Alternating series test is also suitable here.)

So, $-1$ is included in the interval of convergence.

To sum up, the interval of convergence should be $[-1, 1]$.