Math 410. HW 1 Solutions

1. We show the two statements using different approaches.

   (i) Assume, on the contrary, that $\exists r \in \mathbb{Q}$ and $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $r - x \in \mathbb{Q}$. Then $x = r - (r - x) \in \mathbb{Q}$, since $\mathbb{Q}$ satisfies the Field Axioms. This implies $x \in \mathbb{Q}$. Contradiction. Therefore, the sum of a rational and an irrational real numbers is an irrational number.

   (ii) Assume, on the contrary, that $\exists r \in \mathbb{Q}$ and $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $r/x \in \mathbb{Q}$. Let $a, b, c, d \in \mathbb{Z}$ be such that $r = a/b$ and $r/x = c/d$. Then, since $(r/x) \cdot x = r$, $x = \frac{r}{(r/x)}$. Hence, $x = \frac{a/b}{c/d} = \frac{ad}{bc}$ and therefore $r/x \in \mathbb{Q}$. Contradiction.
   Therefore, the quotient of a rational and an irrational real numbers is an irrational number.

2. Assume, on the contrary, that there exists $r \in \mathbb{Q}$ such that $r^2 = 12$. Let $a, b \in \mathbb{Z}, b \neq 0$ such that $r = a/b$ and either a or b (or both) are odd. Then, $12 = r^2 = \frac{a^2}{b^2}$. Hence, $a^2 = 12b^2$. Therefore, a^2 is even, and so a is even and b is odd. Let $c, d \in \mathbb{Z}$ be such that $a = 2c$ and $b = 2d + 1$. Then, $(2c)^2 = 12(2d + 1)^2$. After expanding we obtain $4c^2 = 12(4d^2 + 4d + 1)$, which is equivalent to $c^2 = 12d^2 + 12d + 3$. Since c^2 is odd, c must be odd as well. Let $m \in \mathbb{Z}$ be such that $c = 2m + 1$. Then, we obtain $(2m + 1)^2 = 12d^2 + 12d + 3$. Expanding again, we obtain $4m^2 + 4m + 1 = 12d^2 + 12d + 3$. Therefore, $4m^2 + 4m - 12d^2 - 12d = 3 - 1$. This implies that $2m^2 + 2m - 6d^2 - 6d = 1$, which is impossible since the left hand side is even, whereas the right hand side is odd. Therefore, there exist no rational number r such that $r^2 = 12$.

3. Let $a < b \in \mathbb{Q}$. Let us construct an infinite sequence of distinct irrational numbers $\{x_n\}$ in the interval $[a, b]$. Since irrational numbers are dense in $\mathbb{R}$, there exists some irrational number x such that $a < x < b$. By the Archimedean property, $\exists N \in \mathbb{N}$ such that $b - x < \frac{1}{N}$. Then $b - x < \frac{1}{n+N}$ for all $n \in \mathbb{N}$. Let $x_n = x + \frac{1}{n+N}$. By problem 1(i), $x_n \in \mathbb{R} \setminus \mathbb{Q}$ for all $n \in \mathbb{N}$. Moreover, $\{x_n\}$ is a decreasing sequence satisfying $a < x < x + \frac{1}{n} < x + \frac{1}{N} < b$ for all $n \in \mathbb{N}$. Therefore, there are infinitely many irrational numbers between any two different rational numbers.

4. First, let us note that

$$\sqrt{n^2 + n} - n = (\sqrt{n^2 + n} - n) \cdot \frac{(\sqrt{n^2 + n} + n)}{(\sqrt{n^2 + n} + n)} = \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}. \qquad (*)$$

   Next, we will show that $\lim_{n\to\infty} \sqrt{1 + \frac{1}{n}} = 1$. Let $\epsilon > 0$. We must show that there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $|\sqrt{1 + \frac{1}{n}} - 1| < \epsilon$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \epsilon$. Then, for $n \geq N$, $\frac{1}{n} < \epsilon$ and therefore $1 + \frac{1}{n} < 1 + \epsilon < 1 + 2\epsilon + \epsilon^2 = (1 + \epsilon)^2$. So $\sqrt{1 + \frac{1}{n}} < 1 + \epsilon$ and $\sqrt{1 + \frac{1}{n}} - 1 < \epsilon$. On the other hand, since $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$, we have that $1 + \frac{1}{n} > 1$. Hence $\sqrt{1 + \frac{1}{n}} > 1$ and thus $\sqrt{1 + \frac{1}{n}} - 1 > 0$. Combining, the two steps we get $|\sqrt{1 + \frac{1}{n}} - 1| < \epsilon$ and therefore $\lim_{n\to\infty} \sqrt{1 + \frac{1}{n}} = 1$.

   To conclude, let us make use of the sum and quotient properties of convergent sequences.

$$\lim_{n\to\infty} \sqrt{n^2 + n} - n \overset{(*)}{=} \lim_{n\to\infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} = \frac{1}{\lim_{n\to\infty} \sqrt{1 + \frac{1}{n}} + 1} = \frac{1}{1 + 1} = \frac{1}{2}.$$

5. No. Here is a counterexample. Let $a_n = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$ and $b_n = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$.
   Then, $a_n b_n = 0$ for all $n \in \mathbb{N}$, and therefore $\lim_{n\to\infty} a_n b_n = 0$. However, neither $\lim_{n\to\infty} a_n$ nor $\lim_{n\to\infty} b_n$ exist.