Math 410. HW 1 Solutions

1. We show the two statements using different approaches.

(i) Assume, on the contrary, that \( \exists r \in \mathbb{Q} \) and \( x \in \mathbb{R} \setminus \mathbb{Q} \) such that \( r - x \in \mathbb{Q} \). Then \( x = r - (r - x) \in \mathbb{Q} \), since \( \mathbb{Q} \) satisfies the Field Axioms. This implies \( x \in \mathbb{Q} \). Contradiction. Therefore, the sum of a rational and an irrational real numbers is an irrational number.

(ii) Assume, on the contrary, that \( \exists r \in \mathbb{Q} \) and \( x \in \mathbb{R} \setminus \mathbb{Q} \) such that \( r/x \in \mathbb{Q} \). Let \( a, b, c, d \in \mathbb{Z} \) be such that \( r = a/b \) and \( r/x = c/d \). Then, since \((r/x) \cdot x = r\), \( x = \frac{r/x}{c/d} \). Hence, \( x = \frac{a}{c} \frac{d}{b} \) and therefore \( r/x \in \mathbb{Q} \). Contradiction. Therefore, the quotient of a rational and an irrational real numbers is an irrational number.

2. Assume, on the contrary, that there exists \( r \in \mathbb{Q} \) such that \( r^2 = 12 \). Let \( a, b \in \mathbb{Z} \), \( b \neq 0 \) such that \( r = a/b \) and either \( a \) or \( b \) (or both) are odd. Then \( b^2 \) is odd, and so \( a \) is even and \( b \) is odd. Let \( c, d \in \mathbb{Z} \) be such that \( a = 2c \) and \( b = 2d + 1 \). Then, \((2c)^2 = 12(2d + 1)^2 \). After expanding we obtain \(4c^2 = 12(4d^2 + 4d + 1)\), which is equivalent to \( c^2 = 3d^2 + 3d + 3 \). Since \( c^2 \) is odd, \( c \) must be odd as well. Let \( m \in \mathbb{Z} \) be such that \( c = 2m + 1 \). Then, we obtain \((2m + 1)^2 = 12d^2 + 12d + 3 \). Expanding again, we obtain \(4m^2 + 4m + 1 = 12d^2 + 12d + 3 \). Therefore, \(4m^2 + 4m - 12d^2 - 12d = 3 - 1 \). This implies that \(2m^2 + 2m - 6d^2 - 6d = 1 \), which is impossible since the left hand side is even, whereas the right hand side is odd. Therefore, there exist no rational number \( r \) such that \( r^2 = 12 \).

3. Let \( a < b \in \mathbb{Q} \). Let us construct an infinite sequence of distinct irrational numbers \( \{x_n\} \) in the interval \([a, b]\). Since irrational numbers are dense in \( \mathbb{R} \), there exists some irrational number \( x \) such that \( a < x < b \). By the Archimedean property, \( \exists N \in \mathbb{N} \) such that \( b - x < \frac{1}{N} \). Then \( b - x < \frac{1}{\sqrt{N}} \) for all \( n \in \mathbb{N} \). Let \( x_n = x + \frac{1}{n+\sqrt{N}} \). By problem 1(i), \( x_n \in \mathbb{R} \setminus \mathbb{Q} \) for all \( n \in \mathbb{N} \). Moreover, \( \{x_n\} \) is a decreasing sequence satisfying \( a < x < x + \frac{1}{n} < x + \frac{1}{N} < b \) for all \( n \in \mathbb{N} \). Therefore, there are infinitely many irrational numbers between any two different rational numbers.

4. First, let us note that
\[
\sqrt{n^2 + n} - n = ( \sqrt{n^2 + n} - n ) \frac{\sqrt{n^2 + n} + n}{\sqrt{n^2 + n} + n} = \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{n^2 + n} + n}. \quad (*)
\]

Next, we will show that \( \lim_{n \to \infty} \sqrt{1 + \frac{1}{n}} = 1 \). Let \( \epsilon > 0 \). We must show that there is some \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( |\sqrt{1 + \frac{1}{n}} - 1| < \epsilon \). Let \( N \in \mathbb{N} \) be such that \( \frac{1}{N} < \epsilon \). Then, for \( n \geq N \), \( \frac{1}{n} < \epsilon \) and therefore \( 1 + \frac{1}{n} < 1 + \epsilon < 1 + 2\epsilon + \epsilon^2 = (1 + \epsilon)^2 \). So \( 1 + \frac{1}{n} < 1 + \epsilon \) and \( \sqrt{1 + \frac{1}{n}} < \sqrt{1 + \epsilon} \). On the other hand, since \( \frac{1}{n} > 0 \) for all \( n \in \mathbb{N} \), we have that \( 1 + \frac{1}{n} > 1 \). Hence \( \sqrt{1 + \frac{1}{n}} > 1 \). Combining the two steps we get \( |\sqrt{1 + \frac{1}{n}} - 1| < \epsilon \) and therefore \( \lim_{n \to \infty} \sqrt{1 + \frac{1}{n}} = 1 \).

To conclude, let us make use of the sum and quotient properties of convergent sequences.
\[
\lim_{n \to \infty} \sqrt{n^2 + n} - n \overset{(*)}{=} \lim_{n \to \infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} = \frac{1}{\lim_{n \to \infty} \sqrt{1 + \frac{1}{n}} + 1} = \frac{1}{1+1} = \frac{1}{2}.
\]

5. No. There is a counterexample. Let \( a_n = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases} \) and \( b_n = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \).

Then, \( a_nb_n = 0 \) for all \( n \in \mathbb{N} \), and therefore \( \lim_{n \to \infty} a_nb_n = 0 \). However, neither \( \lim_{n \to \infty} a_n \) nor \( \lim_{n \to \infty} b_n \) exist.