Math 410. HW 2 Solutions

1. Let $\{U_j\}_{j\in\mathbb{N}}$ be an open cover for $A \cup B$. In particular, $\{U_j\}_{j\in\mathbb{N}}$ is an open cover for A and also for B. Since A is compact, there exist $n_1, \ldots, n_k$ such that $A \subset \cup_{i=1}^{k} U_{n_i}$. Similarly, by compactness of B, there exist $m_1, \ldots, m_l$ such that $B \subset \cup_{i=1}^{l} U_{m_i}$. Hence

$$A \cup B \subset \big(\cup_{i=1}^{k} U_{n_i}\big) \cup \big(\cup_{i=1}^{l} U_{m_i}\big),$$

which is a finite subcover of $\{U_j\}_{j\in\mathbb{N}}$. Therefore $A \cup B$ is compact.

2. Not necessarily. Here is an example. Let $A = (0,1], B = \{0\}$. Then $A \cup B = [0,1]$ is compact, since it is a closed and bounded subset of $\mathbb{R}$. However, A is not compact, since it is not closed. The sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is contained in A, but its limit, which is 0, is not.

3. A function $f : (a,b) \to \mathbb{R}$ is not continuous at $x_0 \in (a,b)$ if and only if

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0\ \exists x \text{ such that } |x - x_0| < \delta \text{ but } |f(x) - f(x_0)| \geq \epsilon.$$

4. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{1+e^x}$ is continuous, since it is a composition of continuous functions. Its range equals the interval $(0,1)$ because it is decreasing, $\lim_{x\to-\infty} = 1$ and $\lim_{x\to\infty} = 0$.

5. Let $T > 0$ be such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. Let $J = [-T, 2T]$. Since J is a closed and bounded interval, the restriction of f to J is uniformly continuous (by Theorem 3.17). We will use this fact to show uniform continuity of $f : \mathbb{R} \to \mathbb{R}$.

   Consider sequences $\{x_j\}, \{y_j\} \subset \mathbb{R}$ such that $\lim_{j\to\infty} |x_j - y_j| = 0$. We need to show that $\lim_{j\to\infty} |f(x_j) - f(y_j)| = 0$. Let $u_j = x_j - \lfloor \frac{x_j}{T} \rfloor T$, where $\lfloor x \rfloor$ denotes the integer part of x. Then, $0 \leq u_j < T$ and $f(x_j) = f(u_j)$ by periodicity of f. Now let $v_j = y_j - \lfloor \frac{x_j}{T} \rfloor T$. As above, we have that $f(y_j) = f(v_j)$ by periodicity of f.

   Since $\lim_{j\to\infty} |x_j - y_j| = 0$, there exists $N \in \mathbb{N}$ such that for $j \geq N$, $|x_j - y_j| < T$. Also, since $0 \leq u_j < T$ and $|u_j - v_j| = |x_j - y_j| < T$ for $j \geq N$, by the triangle inequality we have that $-T \leq v_j \leq 2T$ for $j \geq N$. This implies that $u_j, v_j \in J$ for $j \geq N$. Also, $\lim_{j\to\infty} |u_j - v_j| = \lim_{j\to\infty} |x_j - y_j| = 0$. By uniform continuity of f restricted to J, we have that $\lim_{j\to\infty} |f(u_j) - f(v_j)| = 0$. Recalling that $f(x_j) = f(u_j)$ and $f(y_j) = f(v_j)$, this implies that $\lim_{j\to\infty} |f(x_j) - f(y_j)| = 0$. Hence f is uniformly continuous on $\mathbb{R}$.