Math 410. HW 3 Solutions

1. Let $\epsilon > 0$. Let $\delta > 0$ be such that the only point of D in the interval $(x_0 - \delta, x_0 + \delta)$ is x_0. If $x \in D$ and $|x - x_0| < \delta$, then $x = x_0$ and thus $|f(x) - f(x_0)| = 0 < \epsilon$. Then f is continuous at x_0. $\square$

2. From the definition of derivative, $f'(0) = \lim_{y \to 0} \frac{f(y)-f(0)}{y}$. Let $0 < \delta < 1$ be such that whenever $|y| < \delta$, $|\frac{f(y)-f(0)}{y} - f'(0)| < \epsilon/2$.

   Let $x \in (-\sqrt{\delta}, \sqrt{\delta})$. Then $|x^2| < \delta$ and so $|\frac{f(x^2)-f(0)}{x^2} - f'(0)| < \epsilon/2$. Multiplying by $|x|$ and using that $|x| < \delta < 1$ we have that $|\frac{f(x^2)-f(0)}{x} - f'(0)x| < \epsilon/2$. Therefore, by the triangle inequality, $|\frac{f(x^2)-f(0)}{x}| < |f'(0)x| + \epsilon/2$.

   If $f'(0) \neq 0$, let $\tilde{\delta} = \min\{\delta, \frac{\epsilon}{2|f'(0)|}\}$. Then, $|\frac{f(x^2)-f(0)}{x}| < |f'(0)x| + \epsilon/2 \leq \epsilon/2 + \epsilon/2 = \epsilon$. If $f'(0) = 0$ we obtain directly that $|\frac{f(x^2)-f(0)}{x}| < \epsilon/2$. In both cases this implies that $\lim_{x \to 0} \frac{f(x^2)-f(0)}{x} = 0$. $\square$

3. First note that $f(0) = 0$. Then, $\frac{f(h)-f(0)}{h} = \begin{cases} h^2 & \text{if } h \in \mathbb{Q} \\ -h^2 & \text{if } h \notin \mathbb{Q} \end{cases}$.

   Therefore $\lim_{h \to 0} \frac{f(h)-f(0)}{h} = 0$ so f is differentiable at 0, and $f'(0) = 0$. $\square$

4. We will show that $f'(x) = 0$ for all $x \in \mathbb{R}$. Assume this is not the case and let $x_0 \in \mathbb{R}$ be such that $f'(x_0) \neq 0$.

   First suppose $f'(x_0) > 0$. Since $f''(x) \geq 0$ for all x, we know that f' is non-decreasing. In particular, for every $x \geq x_0$ we have $f'(x) \geq f'(x_0) > 0$. Let $x_1 = x_0 + \frac{1-f(x_0)}{f'(x_0)}$. We claim that $f(x_1) \geq 1$. Indeed, since $f(x_0) \leq 0$ and $f'(x_0) > 0$, $x_1 > x_0$. By the Mean Value Theorem, the exists some $z \in [x_0, x_1]$ such that $f'(z) = \frac{f(x_1)-f(x_0)}{x_1 - x_0}$. Therefore,

   $$f(x_1) = f(x_0) + f'(z)(x_1 - x_0) = f(x_0) + f'(z)(\frac{1-f(x_0)}{f'(x_0)}) \geq f(x_0) + f'(x_0)(\frac{1-f(x_0)}{f'(x_0)}) = 1.$$

   This contradicts the assumption that $f(x) \leq 0$ for all $x \in \mathbb{R}$.

   We can apply the same reasoning to show that there is no x_0 such that $f'(x_0) < 0$. Suppose $f'(x_0) < 0$. Using that f' is non-decreasing, we know that for every $x \leq x_0$ we have $f'(x) \leq f'(x_0) < 0$. Let $x_1 = x_0 + \frac{1-f(x_0)}{f'(x_0)}$. We claim that $f(x_1) \geq 1$. Indeed, since $f(x_0) \leq 0$ and $f'(x_0) < 0$, $x_1 < x_0$. By the Mean Value Theorem, the exists some $z \in [x_1, x_0]$ such that $f'(z) = \frac{f(x_1)-f(x_0)}{x_1 - x_0}$. Therefore,

   $$f(x_1) = f(x_0) + f'(z)(x_1 - x_0) = f(x_0) + f'(z)(\frac{1-f(x_0)}{f'(x_0)}) \geq f(x_0) + f'(x_0)(\frac{1-f(x_0)}{f'(x_0)}) = 1.$$

   This contradicts the assumption that $f(x) \leq 0$ for all $x \in \mathbb{R}$. Hence, there is no x_0 such that $f'(x_0) < 0$.

   Combining the two parts, we conclude that $f'(x) = 0$ for all $x \in \mathbb{R}$. This implies that f is constant. $\square$

5. Let $m, M \in \{1, \ldots, k\}$ be such that

   $$f(x_m) = \min\{f(x_1), \ldots, f(x_k)\} \text{ and } f(x_M) = \max\{f(x_1), \ldots, f(x_k)\}.$$

   Then, $f(x_m) \leq \frac{f(x_1)+\cdots+f(x_k)}{k} \leq f(x_M)$. By the Intermediate Value Theorem, there exists a point $z \in [\min\{x_m, x_M\}, \max\{x_m, x_M\}] \subset [a, b]$ such that $f(z) = \frac{f(x_1)+\cdots+f(x_k)}{k}$. $\square$