MATH 630, Fall 2011, Extra HW

1) [10 extra points] Establish the relationships between:

a) the notion of absolute continuity as defined in Definition 3.3.10.a for the measure space $(X, \mathcal{Z}, \mu) = ([a,b], \mathcal{B}([a,b]), m|_{[a,b]})$

b) the notion of absolute continuity defined in Definition 4.5.2.

2) [10 extra points] Provide a generalization of the notion of absolute continuity defined in Definition 4.5.2 to $[a,b]\times[c,d] \subset \mathbb{R}^2$, etsablish its properties and relationship to the notion of absolute continuity as defined in Definition 3.3.10.a.

3) [10 extra points] Consider the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and a function $f : \mathbb{R} \to \mathbb{R}$, right continuous at each point. Then, $f \in BV(\mathbb{R})$ if and only if

$$\mu_f((a,b]) = f(b) - f(a) \tag{1}$$

can be uniquely extended to an element of $SM(\mathbb{R})$, which is bounded. In this situation f is continuous at x_0 if and only if $\mu_f(\{x_0\}) = 0$.

4) [20 extra points] Find a correct formulation for Theorem 5.4.1 that characterizes $BV_{loc}(\mathbb{R})$.

5) [10 points] Prove the following inclusions and find examples which show that they are proper:

$$\mathrm{Lip}([a,b]) \subset AC([a,b]) \subset BV([a,b]).$$

6) [10 points] Let $f : [a,b] \to \mathbb{R}$ satisfy the following condition:

$$\forall\, \epsilon > 0 \quad \exists\, \delta > 0, \quad \text{such that} \quad \left| \sum_{j=0}^{n} (f(y_j) - f(x_j) \right| \leq \epsilon,$$

for every finite collection of intervals $(x_j, y_j) \subset [a,b]$, $j = 1, \ldots, n$, and $\sum_{j=1}^{n}(y_j - x_j) < \delta$. Prove that f is locally Lipschitz on $[a,b]$.

7) [10 points] Prove that the following are equivalent:

a) $f \in AC([a,b])$;

b) $f : [a,b] \to \mathbb{R}$ satisfies:

$$\forall\, \epsilon > 0 \quad \exists\, \delta > 0, \quad \text{such that} \quad \left| \sum_{j=0}^{n} (f(y_j) - f(x_j) \right| \leq \epsilon,$$

for every finite collection of disjoint intervals $(x_j, y_j) \subset [a, b]$, $j = 1, \ldots, n$, and $\sum_{j=1}^{n}(y_j - x_j) < \delta$.

8) [10 points] Show that in the definition of absolute continuity on $[a, b]$ one cannot drop the requirement that the intervals (x_j, y_j) are pairwise disjoint.