MATH 648 Q, HW 1

1) Show that the limit of the ratio of D-dimensional volumes of a cube of side $2r$ and a ball of radius r inscribed in the cube, is 0 when $r > 0$ is fixed and $D \to \infty$.

Note that the cube is in fact a ball in $\mathbb{R}^D$ equipped with the ℓ^∞ metric (i.e., $\|x\|_\infty = \max(|x_1|, \ldots, |x_D|)$), compared to the standard ball being a ball in the Euclidean metric ℓ^2 ($\|x\|_2^2 = \sum_{j=1}^D |x_j|^2$).

What is the limit of the ratio of volumes of balls in ℓ^2 and ℓ^1 metrics, respectively, when $r > 0$ is fixed and $D \to \infty$? (Here $\|x\|_1 = \sum_{j=1}^D |x_j|$.)

What about the ratios of volumes of balls in any ℓ^p and ℓ^q metrics?

2) Show that the ratio of the volume of an ϵ spherical shell of radius r (i.e., volume of the set of $x \in \mathbb{R}^D$ for which $r(1-\epsilon) \leq \|x\|_2 \leq r$) and the volume the ball of radius r converges to 1 when $r > 0$ is fixed and $D \to \infty$.

What can you say about the limits of such ratios, when ℓ^2 norm is replaced with ℓ^∞ and ℓ^1, respectively?