Laplace transform

Let f be a function of one real variable. Define formally the following integral:

$$\mathcal{L}_2(f)(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt,$$

where $s \in \mathbb{C}$. $\mathcal{L}_2(f)$ is called the *two-sided Laplace transform* of f. A sufficient condition for the existence of the Laplace transform $\mathcal{L}_2(f)(s)$ is that the integral

$$\int_{\mathbb{R}} |f(t)|e^{-\Re(s)t}dt < \infty.$$

Example 0.1. Let

$$f(t) = \begin{cases} 1, & t > 0, \\ e^t, & t \leq 0. \end{cases}$$

Then, for $s \in \mathbb{R}$,

$$\mathcal{L}_2(f)(s) = \int_{-\infty}^{0} e^{(1-s)t}dt + \int_{0}^{\infty} e^{-st}dt.$$

Clearly, the first integral converges for $s < 1$, while the second integral exists for $s > 0$. Thus, $\mathcal{L}_2(f)(s)$ exists for $s \in (0, 1)$.

The *one-sided Laplace transform* $\mathcal{L}(f)$ is defined as:

$$\mathcal{L}(f)(s) = \int_{0}^{\infty} f(t)e^{-st}dt,$$

where, formally, $s \in \mathbb{C}$.

We say that a piecewise continuous function f is of *exponential type* s_0 if there exists $s_0 \in \mathbb{R}$ such that for all $s > s_0$:

$$\lim_{t\to\infty} f(t)e^{-st} = 0.$$

Lemma 0.2. If f is a piecewise continuous function of exponential type s_0, then the integral which defines the one-sided Laplace transform of f, $\mathcal{L}(f)$, converges for all $s \in \mathbb{C}$ such that $\Re(s) > s_0$. Moreover, this integral converges uniformly for all s such that $\Re(s) > s_1$ for some (any) $s_1 > s_0$.

Theorem 0.3. Let f be a piecewise continuous function, for which the integral

$$\int_0^\infty f(t)e^{-s_0 t}dt$$

converges for some $s_0 \in \mathbb{R}$. Then, the Laplace transform $\mathcal{L}(f)(s)$ converges for all $s \in \mathbb{C}$ such that $\Re(s) > s_0$. Moreover, the Laplace transform converges uniformly for $\{s \in \mathbb{C} : |\angle(s; s_0)| \leq \theta < \pi/2\}$ for some (any) θ.

Example 0.4. We cannot get the convergence for the case $\Re(s) = s_0$. Indeed, consider the function:

$$f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ \frac{1}{t}, & t \leq 1. \end{cases}$$

Then, for $s = iu$, $u \in \mathbb{R}$,

$$\int_1^\infty \frac{e^{-iut}}{t}dt = \int_1^\infty \frac{\cos(ut)}{t}dt - i\int_1^\infty \frac{\sin(ut)}{t}dt$$

converges, but

$$\int_1^\infty \frac{e^0}{t}dt = \int_1^\infty \frac{1}{t}dt = \infty.$$

The two-sided Laplace transform may be written as:

$$\mathcal{L}_2(f)(s) = \int_{-\infty}^0 f(-t)e^{st}dt + \int_0^\infty f(t)e^{-st}dt = \mathcal{L}(f)(s) + L(\tilde{f})(-s),$$

where $\tilde{f}(t) = f(-t)$. Suppose that $\mathcal{L}(f)(s)$ converges for s such that $\Re(s) > \alpha$. On the other hand, assume that $\mathcal{L}(\tilde{f})(s)$ converges for s such that $\Re(s) > \beta$. Then, $L(\tilde{f})(-s)$ converges for s such that $-\Re(s) = \Re(-s) > \beta$, i.e., $\Re(s) < -\beta$.

These calculations imply that $\mathcal{L}_2(f)(s)$ converges for all s which satisfy $\alpha < \Re(s) < -\beta$. In particular, if $-\beta < \alpha$, $\mathcal{L}_2(f)$ *never* converges.

Example 0.5. Consider the following function

$$f(t) = \begin{cases} e^{at}, & t \leq 0, \\ e^{bt}, & t \geq 0, \end{cases}$$

where $b > a$. Then

$$\mathcal{L}_2(f)(s) = \int_{-\infty}^{0} e^{(a-s)t} dt + \int_{0}^{\infty} e^{(b-s)t} dt.$$

The second integral converges only if $(b - s) < 0$, that is for $s > b$. In particular, $s > a$. But this implies that the first integral diverges.

Theorem 0.6. The Laplace transform is a linear operation:

$$\mathcal{L}_2(f + \alpha g)(s) = \mathcal{L}_2(f)(s) + \alpha \mathcal{L}_2(g)(s)$$

and

$$\mathcal{L}(f + \alpha g)(s) = \mathcal{L}(f)(s) + \alpha \mathcal{L}(g)(s).$$

Example 0.7.

$$\mathcal{L}(1)(s) = \frac{1}{s}$$

$$\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(\sqrt{t})(s) = \frac{\sqrt{\pi}}{2\sqrt{s^3}}$$

$$\mathcal{L}(\cos(bt))(s) = \frac{s}{s^2 + b^2}$$

$$\mathcal{L}(\sin(bt))(s) = \frac{b}{s^2 + b^2}$$

Theorem 0.8. Let f and its derivatives by continuous functions of exponential type. Then

$$\mathcal{L}(f^{(n)})(s) = s^n \mathcal{L}(f)(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0).$$

Theorem 0.9. Let f be a continuous piecewise function. Then

$$\mathcal{L}\left(\int_0^t f(\tau) d\tau \right)(s) = \frac{\mathcal{L}(f)(s)}{s}.$$

The two-sided Laplace transform may be viewed as closely connected to the Fourier transform:

$$\begin{aligned}\mathcal{L}_2(f)(s) &= \int_{\mathbb{R}} f(t)e^{-st}dt = \int_{\mathbb{R}} f(t)e^{-\Re(s)t}e^{-i\Im(s)t}dt \\ &= \int_{\mathbb{R}} f(t)e^{-\Re(s)t}e^{-2\pi i\Im(s)t/2\pi}dt \\ &= \mathcal{F}(f(\cdot)e^{-\Re(s)\cdot})(\Im(s)/2\pi).\end{aligned}$$

This observation allows us to study many properties which were true for the Fourier Transform and which, we hope, will be satisfied by the Laplace transforms, as well. For example, consider a convolution $f*g$. Let s be such that all the involved integrals and transforms converge. Then,

$$\begin{aligned}\mathcal{L}_2(f*g)(s) &= \mathcal{F}(f*g(t)e^{-\Re(s)t})(\Im(s)/2\pi) \\ &= \mathcal{F}\left(\int_{-\infty}^{\infty} f(\tau)g(t-\tau)e^{-\Re(s)t}d\tau\right)(\Im(s)/2\pi) \\ &= \mathcal{F}\left(\int_{-\infty}^{\infty} f(\tau)e^{-\Re(s)\tau}g(t-\tau)e^{-\Re(s)(t-\tau)}d\tau\right)(\Im(s)/2\pi) \\ &= \mathcal{F}\left(f(\tau)e^{-\Re(s)\tau} * g(t-\tau)e^{-\Re(s)(t-\tau)}\right)(\Im(s)/2\pi) \\ &= \mathcal{F}\left(f(\tau)e^{-\Re(s)\tau}\right)(\Im(s)/2\pi)\cdot\mathcal{F}\left(g(t-\tau)e^{-\Re(s)(t-\tau)}\right)(\Im(s)/2\pi) \\ &= \mathcal{L}_2(f)(s)\cdot\mathcal{L}_2(g)(s).\end{aligned}$$

For the case of one-sided Laplace transform we may write:

$$\mathcal{L}(f)(s) = \mathcal{L}_2(\chi_{[0,\infty)}f)(s).$$

Thus,

$$\begin{aligned}\mathcal{L}(f)(s)\cdot\mathcal{L}(g)(s) &= \mathcal{L}_2((\chi_{[0,\infty)}f)*(\chi_{[0,\infty)}g))(s) \\ &= \mathcal{L}_2\left(\int_{\mathbb{R}} f(\tau)\chi_{[0,\infty)}(\tau)g(t-\tau)\chi_{[0,\infty)}(t-\tau)\right)(s) \\ &= \mathcal{L}\left(\int_0^t f(\tau)g(t-\tau)d\tau\right)(s).\end{aligned}$$

The above formula defines the natural convolution for the one-sided Laplace transfrom.