MATH 141, FALL 2014, **Volume of a cone**

Consider the problem of computing volume of a cone of height h and base with radius R. Naturally, you know from school that its volume is:

$$V = \frac{1}{3}\pi R^2 h.$$

There are, however, several ways of establishing this fact.

1) In class, we used the definition of volume (Definition 6.1) to do this computation. Let us recall what we did, for completeness.

We positioned the cone parallel to the x-axis, so that its base was at $x = 0$ and apex at $x = h$. The volume formula from Definition 6.1 can thus be rewritten as:

$$V = \int_0^h A(x)\,dx,$$

where $A(x)$ represents the cross-sectional area (perpendicular to the x-axis) of the cone at x. These cross-sections are discs or radius r. The radius r depends on x (hence we write $r = r(x)$), and therefore $A(x) = \pi r^2$. We then used the similarity of triangles to establish that:

$$r = \frac{R(h-x)}{h}.$$

Plugging this into the formula for $A(x)$ we obtain:

$$\begin{aligned} V &= \int_0^h \frac{\pi(R(h-x))^2}{h^2}\,dx = \frac{\pi R^2}{h^2}\int_0^h (h-x)^2\,dx \\ &= \frac{\pi R^2}{h^2}\int_0^h (h^2 - 2hx + x^2)\,dx \qquad (1) \\ &= \frac{\pi R^2}{h^2}\left(h^2 x - hx^2 + \frac{x^3}{3}\right|_0^h = \frac{\pi R^2 h}{3}. \end{aligned}$$

2) Another approach would be to reposition the cone with apex at the origin, but still parallel to the x-axis. Clearly this does not affect the volume formula, not the generic formula for the cross-sectional areas. However, what changes is the relationship between similar triangles. In this new situation we have the following proportion:

$$\frac{r}{x} = \frac{R}{h},$$

which leads to $r = xR/h$. Plugging this into the formula for $A(x)$ we obtain:

$$\begin{aligned} V &= \int_0^h \frac{\pi(Rx)^2}{h^2}\,dx \\ &= \frac{\pi R^2}{h^2}\int_0^h x^2\,dx \\ &= \frac{\pi R^2}{h^2}\left(\frac{x^3}{3}\right|_0^h = \frac{\pi R^2 h}{3}. \end{aligned} \tag{2}$$

3) We can also use some of the methods introduced in class to compute this volume. We shall start with the "disc method":

$$V = \int_0^h \pi(f(x)^2)\,dx. \tag{3}$$

For this, we again position the cone parallel to the x-axis.

3a) We will first consider the positioning with the base of the cone at $x = 0$ and apex at $x = h$ (i.e., as was in part 1). In this case the formula for a linear function $f(x)$, which passes through points $(0, R)$ and $(h, 0)$ is:

$$f(x) = -\frac{R}{h}x + R.$$

Plugging this into (3), we get:

$$V = \int_0^h \pi\left(\frac{R}{h}x - R\right)^2 dx = \pi\frac{R^2}{h^2}\int_0^h (x-h)^2\,dx.$$

It is easy to see that the above is the same calculation as in (1), since $(x-h)^2 = (h-x)^2$.

3b) When we position the cone with the apex at $x = 0$ and base at $x = h$ (i.e., as was in part 2), what changes is we need to find a linear function $f(x)$ that passes through points $(0,0)$ and (h,R). This leads us to obtain:

$$f(x) = \frac{R}{h}x.$$

Plugging this into (3), we get:

$$V = \int_0^h \pi \left(\frac{R}{h}x\right)^2 \, dx,$$

and we close by noting that this is the same computation as in (2).

4) The final method to use for this problem is the "shell method":

$$V = \int_0^R 2\pi x(f(x) - g(x))\, dx. \tag{4}$$

This is the method where the functions used to describe the solid are dependent on x, but the rotation is with respect to the y-axis. Moreover, we use R as the upper limit, as the cone shall be now in vertical position (!).

4a) To begin, we shall position the cone parallel to the y-axis, with the base of the cone at $y = 0$ and apex at $y = h$. We easily observe that in this case $g(x) = 0$. The function $f(x)$ is described as linear function that passes through points $(0,h)$ and $(R,0)$. Thus, its formula is:

$$f(x) = -\frac{h}{R}x + h.$$

Plugging this into (4), we get:

$$V = \int_0^R 2\pi x \left(-\frac{h}{R}x + h\right) dx = 2\pi \left(-\frac{h}{3R}x^3 + \frac{hx^2}{2}\right|_0^R = 2\pi(-\frac{hR^2}{3} + \frac{hR^2}{2}) = \frac{1}{3}\pi R^2 h.$$

4b) We could also position the cone parallel to the y-axis, with the apex of the cone at $y = 0$ and base at $y = h$. In such case $f(x) = h$ (a constant function), and $g(x)$ must be a linear function passing through points $(0,0)$ and (R,h). Thus, its formula is:

$$g(x) = \frac{h}{R}x.$$

Plugging this into (4), we get:

$$V = \int_0^R 2\pi x(h - \frac{h}{R}x)\,dx = 2\pi \left(h\frac{x^2}{2} - \frac{hx^3}{3R} \right|_0^R = \frac{1}{3}\pi R^2 h.$$

5) Last but not least, if the cone is positioned vectically (i.e., parallel to the y-axis) and you would prefer to use the "disc method", just re-label the axes: interchanging the roles of x and y. Doing this with the cone that has its base at $y = 0$ corresponds to solving part 1). And if the cone has its apex at $y = 0$ then this interchange leads you to part 2).