MATH 141, FALL 2015, Sample Differentiation Probelms

Differentiation for most of Calculus 2 students seems like a piece of cake: you already know it very well, you have studied many examples in Calculus 1, and it seems like there are no more surprises there. However, it is not really the case. As we continue our venture into Calculus, we are discovering new layers of difficulties. Including in differentiation. Take, for example the following function:

$$f(x) = x^x.$$

Depending on your preference, you might want to treat it as a type of a power function or exponential function. One treats the base as the variable, the other exponent. Either approach, however, introduces the difficulty of dealing with the other component. The only proper way to resolve this difficulty is to use the definition of generalized exponential:

$$f(x) = x^x = e^{x\ln(x)}.$$

Only now we see how to proceed. First, by the Chain Rule we get:

$$f'(x) = \frac{d}{dx}\left(e^{x\ln(x)}\right) = \left(\frac{d}{dx}e^x\bigg|_{x\ln(x)}\right)\frac{d}{dx}\left(x\ln(x)\right).$$

Next, using the Product Rule and formulas for the derivatives of natural exponential function and natural logarithmic function, we obtain:

$$f'(x) = e^{x\ln(x)}\left(\ln(x)+1\right) = (1+\ln(x))x^x.$$

Some other examples of this type:

1)

$$f(x) = \sin(x)^{\cos(x)} = e^{\cos(x)\ln(\sin(x))}.$$

Hence

$$\begin{aligned} f'(x) &= e^{\cos(x)\ln(\sin(x))}\frac{d}{dx}\left(\cos(x)\ln(\sin(x))\right) \\ &= \sin(x)^{\cos(x)}\left[-\sin(x)\ln(\sin(x)) + \cot(x)\cos(x)\right]. \end{aligned}$$

2)

$$f(x) = \sqrt{1-x^2}^{\ln(x)} = e^{\ln(x)\ln(\sqrt{1-x^2})}.$$

Hence,

$$\begin{aligned} f'(x) &= e^{\ln(x)\ln(\sqrt{1-x^2})}\frac{d}{dx}\left(\ln(x)\ln(\sqrt{1-x^2})\right) \\ &= \sqrt{1-x^2}^{\ln(x)}\left(\frac{\ln(\sqrt{1-x^2})}{x} + \frac{-2x\ln(x)}{2\sqrt{1-x^2}\sqrt{1-x^2}}\right) \\ &= \sqrt{1-x^2}^{\ln(x)}\left(\frac{\ln(\sqrt{1-x^2})}{x} - \frac{x\ln(x)}{1-x^2}\right). \end{aligned}$$

Do note, however, that there is another way to represent this function:

$$f(x) = \sqrt{1-x^2}^{\ln(x)} = \left((1-x^2)^{1/2}\right)^{\ln(x)} = (1-x^2)^{\frac{1}{2}\ln(x)} = e^{\frac{1}{2}\ln(x)\ln(1-x^2)}.$$

This leads to:

$$\begin{aligned} f'(x) &= \sqrt{1-x^2}^{\ln(x)}\frac{d}{dx}\left(\frac{1}{2}\ln(x)\ln(1-x^2)\right) \\ &= \frac{1}{2}\sqrt{1-x^2}^{\ln(x)}\left(\frac{\ln(1-x^2)}{x} - \frac{2x\ln(x)}{1-x^2}\right). \end{aligned}$$

Please note that both solutions are equal, as

$$\ln(\sqrt{1-x^2}) = \frac{1}{2}\ln(1-x^2),$$

but the 2nd approach may be considered easier, as the differentiation is more direct.