MATH 416, extra problems
1. Suppose $a$ divides $b$ and $b$ divides $a$. Must $a = b$?
2. Suppose $a$ divides $b$ and $b$ divides $c$. Must $a$ divide $c$?
3. Suppose $a^2$ divides $b^2$. Must $a$ divide $b$?
4. Prove that if $p^2 \in \mathbb{Z}$ is a prime number, then $\sqrt{p}$ is not a rational number.
5. Let $N > 0$ be given and suppose $x, y \in \mathbb{R}^N$ satisfy $\|x\|_2 = \|y\|_2 = 1$.
   a) Find $x$ such that $\|x\|_1$ is as large as possible.
   b) Find $y$ such that $\|y\|_\infty$ is as small as possible.
6. Prove that $\|x - y\| \geq \|\|x\| - \|y\||$ for any vectors $x, y$ in a normed vector space $X$.
7. Find the 1-periodization of the function $f(x) = e^{-|x|}$.
8. For real $\epsilon > 0$ and $\alpha$, define the dilation operator $D_\epsilon$ and the translation operator $T_\alpha$, which act on functions $f = f(t)$ of one real variable as follows:
   $$T_\alpha(u)(t) = u(t - \alpha) \quad D_\epsilon(u)(t) = \epsilon^{-1/2}u(t/\epsilon).$$
   a) Show that these are linear transformations with inverses $T_\alpha^{-1} = T_{-\alpha}$ and $D_\epsilon^{-1} = D_{1/\epsilon}$
   b) Compute the composition $T_\alpha(D_\epsilon(F))$ for a function $F = F(x)$.
9. Show that the set of functions $\{1, \sqrt{2}\sin(2\pi nt), \sqrt{2}\cos(2\pi nt) : n = 1, 2, 3, \ldots\}$ is orthonormal with respect to the Hermitean inner product.
10. Show that the set of functions $\{\sqrt{2}\sin(2\pi nt) : n = 1, 2, 3, \ldots\}$ is orthonormal with respect to the real inner product.
11. Show that the set of functions $\{1, \sqrt{2}\cos(2\pi nt) : n = 1, 2, 3, \ldots\}$ is orthonormal with respect to the real inner product.
12. Compute the sine-cosine Fourier series of the 1-periodic function $f(x) = \cos^2(2\pi x)$.
13. Compute the complex exponential Fourier series of the 1-periodic function $\sin(2\pi kt - d)$, where $d$ is a constant real number, and $k$ is an integer.
14. Write out explicitly the matrices for the $2 \times 2$ and $4 \times 4$ Discrete Fourier and Hartley transforms ($F_2, F_4, H_2$ and $H_4$).
15. Write out explicitly the matrices for the $1 \times 1$ Discrete Fourier, Hartley, and DCT-IV transforms ($F_1$, $H_1$ and $CIV_1$).

16. Compute the complex exponential Fourier coefficients of the function $e^{j\alpha x} f(x)$, $x \in [0, 1]$, in terms of the complex exponential Fourier coefficients of the function $f(x)$. 