

MATH 141, FALL 2014, **Two examples of improper integrals**

Here are some examples of things which may go wrong with improper integrals.

A) Consider the following integral

$$I = \int_{-\infty}^{+\infty} \frac{1}{(x-1)^2} dx.$$

It is not difficult to see how to integrate the associated indefinite integral by (almost) directly taking an antiderivative:

$$\int \frac{1}{(x-1)^2} dx = -\frac{1}{x-1} + C.$$

If you thought of using this antiderivative without understanding that your integral I is an improper integral, you would get something like:

$$I = \left(-\frac{1}{x-1} \Big|_{-\infty}^{+\infty} = \lim_{x \rightarrow +\infty} -\frac{1}{x-1} - \lim_{x \rightarrow -\infty} -\frac{1}{x-1} = 0 - 0 = 0. \right.$$

This, however, is incorrect, as we failed to observe that there is a vertical asymptote at $x = 1$ for our function under the integral, and thus the definition of the improper integral requires us to isolate these points away from each other. As such we must consider the following:

$$\int_{-\infty}^0 \frac{1}{(x-1)^2} dx + \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx + \int_2^{+\infty} \frac{1}{(x-1)^2} dx = A+B+C+D.$$

If each and every of the above 4 integrals converges, then I converges, and we can ask the question of what does I converge to. Otherwise, even if only 1 of the above 4 integrals diverges, I diverges. Thus, we analyze them separately, utilizing the antiderivative we already found:

$$A = \lim_{c \rightarrow -\infty} -\frac{1}{x-1} \Big|_c^0 = \lim_{c \rightarrow -\infty} 1 + \frac{1}{c-1} = 1.$$

$$B = \lim_{c \rightarrow 1^-} -\frac{1}{x-1} \Big|_0^c = \lim_{c \rightarrow 1^-} -\frac{1}{c-1} - 1 = +\infty.$$

$$C = \lim_{c \rightarrow 1^+} \left. -\frac{1}{x-1} \right|_c^2 = \lim_{c \rightarrow 1^+} -1 + \frac{1}{c-1} = +\infty.$$

$$D = \lim_{c \rightarrow +\infty} \left. -\frac{1}{x-1} \right|_2^c = \lim_{c \rightarrow +\infty} -1 + \frac{1}{c-1} = -1.$$

As A and D are both divergent, so is I .

Of course, it suffices to notice that just one of the 4 integrals is divergent, to make the same conclusion about I .

B) Consider the following integral

$$I = \int_1^{+\infty} \frac{1}{x(x+1)} dx.$$

Because

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1},$$

one is tempted to write

$$I = \int_1^{+\infty} \frac{1}{x} dx - \int_1^{+\infty} \frac{1}{x+1} dx = \ln(|x|)|_1^{+\infty} - \ln(|x+1|)|_1^{+\infty}.$$

As logarithm diverges at infinity, either of the integrals on the right hand side is divergent, and one is tempted to conclude the same about the original integral I . However, since $\infty - \infty$ is an indeterminate form, we cannot use standard arithmetic operations to it, as we cannot interpret the results of such operations.

Therefore, we cannot interpret the left and right hand sides of the above equation as equal. (In a manner similar to being unable to write $1 = (1 + \infty) - \infty$.) As a matter of fact, it is easy to see that

$$\frac{1}{x(x+1)} \leq \frac{1}{x^2},$$

and, since we know that

$$\int_1^{+\infty} \frac{1}{x^2} dx$$

is convergent, by the Comparison Theorem, we also conclude that I is convergent.