SOLUTION TO PROBLEM 5 ON MIDTERM 1

You can start like this:

\[
\lim_{x \to \infty} \left( 1 + \frac{1}{x^2} \right)^{x^2} = (2 \text{ points}) \lim_{x \to \infty} e^{x^2 \ln(1+1/x^2)} = (2 \text{ points}) e^{\lim_{x \to \infty} x^2 \ln(1+1/x^2)}.
\]

We now consider \( \lim_{x \to \infty} (x^2 \ln(1 + 1/x^2)):\)

\[
\lim_{x \to \infty} x^2 \ln \left( 1 + \frac{1}{x^2} \right) = (2 \text{ points}) \lim_{x \to \infty} \frac{\ln(1 + 1/x^2)}{1/x^2}
\]

Next observe that de l’Hopital’s rule applies to the limit above on the right. Thus, by applying it we have:

\[
\lim_{x \to \infty} \frac{\ln(1 + 1/x^2)}{1/x^2} = \lim_{x \to \infty} \frac{1}{1+1/x^2} \cdot \frac{(-2)}{x^3}
\]

Each correctly computed derivative is worth 4 points (for the total of 8 points). I take away 2 points for each incorrect constant in the computation of either of the derivatives. Further, we notice that

\[
\lim_{x \to \infty} \frac{1}{1+1/x^2} \cdot \frac{(-2)}{x^3} = \lim_{x \to \infty} \frac{1}{1 + 1/x^2} = 1,
\]

which is also worth 4 points.

Last, we plug this observation into the original problem to obtain:

\[
\lim_{x \to \infty} \left( 1 + \frac{1}{x^2} \right)^{x^2} = e^{\lim_{x \to \infty} x^2 \ln(1+1/x^2)} = e^1 = e. \quad (2 \text{ points})
\]