

Problem 1

Find the length L of the curve given parametrically by:

$$\begin{cases} x = \sin t - t \cos t \\ y = t \sin t + \cos t \end{cases} \quad 0 \leq t \leq \pi.$$

What is the starting point of this curve? What is the ending point of this curve? Find at least one intermediate point and sketch the graph of the curve.

Solution:

Recall the formula for the length of a curve given parametrically:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \quad (2)$$

Here we have

$$\begin{aligned} \frac{dx}{dt} &= \cos t + t \sin t - \cos t \\ &= t \sin t, \end{aligned} \quad (3)$$

and

$$\begin{aligned} \frac{dy}{dt} &= -\sin t + t \cos t + \sin t \\ &= t \cos t. \end{aligned} \quad (3)$$

Then

$$\begin{aligned} L &= \int_0^\pi \sqrt{(t \sin t)^2 + (t \cos t)^2} dt \\ &= \int_0^\pi \sqrt{t^2 (\sin^2 t + \cos^2 t)} dt \\ &= \int_0^\pi t dt = \frac{t^2}{2} \Big|_0^\pi = \frac{\pi^2}{2}. \end{aligned} \quad (7)$$

The starting point is given by $t = 0$:

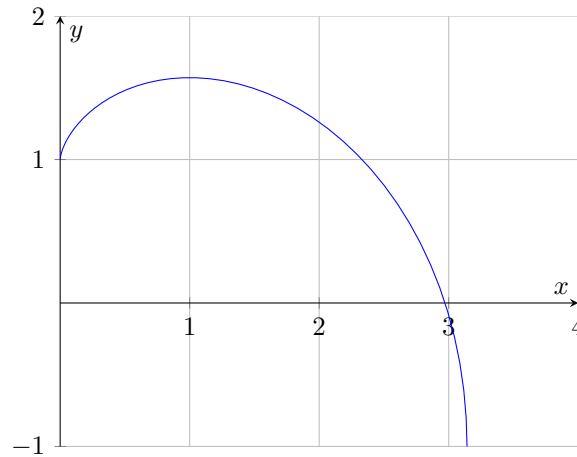
$$\begin{cases} x = \sin 0 - 0 \cos 0 = 0, \\ y = 0 \sin 0 + \cos 0 = 1, \end{cases} \Rightarrow \text{the starting point is } (0, 1). \quad (3)$$

The ending point is given by $t = \pi$:

$$\begin{cases} x = \sin \pi - \pi \cos \pi = \pi, \\ y = \pi \sin \pi + \cos \pi = -1, \end{cases} \Rightarrow \text{the end point is } (\pi, -1). \quad (3)$$

Any intermediate point will give a good indication of what the curve should look like. For example, at $t = \frac{\pi}{2}$ the point is $(1, \frac{\pi}{2})$.

The sketch of the curve should look roughly like this:



(4)