Midterm 4, Problem 1

December 11, 2013

Problem (25pts): Determine whether the following series converge (absolutely/conditionally). If they converge, find their value.

1. $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k 5^{k-1}}{3^{2k}}$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$

Solution:

Part 1. 15 points.

(2pts): We have

$$\sum_{k=2}^{\infty} \frac{(-1)^k 5^{k-1}}{3^{2k}} = \frac{1}{5} \sum_{k=2}^{\infty} \left(\frac{-5}{9}\right)^k$$

(6pts): This is a geometric series with ratio $r = \dfrac{-5}{9}$ and first term $a = \dfrac{5}{81}$. [3 points given for each item.]

(3pts): Thus, since $|r| < 1$, the series **converges absolutely**.

(4pts): In particular, it converges to

$$\frac{a}{1-r} = \frac{\frac{5}{81}}{1 - \left(-\frac{5}{9}\right)} = \boxed{\frac{5}{126}}$$

(Note: Students may alternatively use the root or ratio tests to determine convergence of the series. Only doing this, without finding the value of the series, will merit **8** points.)

Part 2. 10 points.

(2pts): We use the Integral Test to determine the behavior of the series.

(2pts): Let $f(x) = \frac{1}{x \cdot \ln(x)}$. Then, f is decreasing, integrable, and agrees with the sequence that we are summing. Thus, the original sum converges iff $\displaystyle\int_{x=2}^{\infty} \frac{\mathrm{dx}}{x \cdot \ln(x)}$ converges.

(5pts): Now,

$$\begin{aligned}\int_{x=2}^{\infty} \frac{\mathrm{dx}}{x \cdot \ln(x)} &= \lim_{b\to\infty} \left(\int_{x=2}^{b} \frac{\mathrm{dx}}{x \cdot \ln(x)} \right) \\ &= \lim_{b\to\infty} \left(\ln\ln b - \ln\ln 2 \right) \\ &= +\infty, \quad \text{i.e. diverges.}\end{aligned}$$

(1pts): Thus, the original sum $\boxed{\textit{diverges.}}$