*#2.* Calculate the integral $\int \frac{x^2+4}{x(x-1)^2}\,dx$.

*Solution.* This will be straightforward if we can rewrite the integrand in a simpler form. We'll use the partial fractions decomposition

$$\frac{x^2+4}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2},$$

and now we'll solve for the unknown constants $A$, $B$, and $C$. By multiplying both sides by $x(x-1)^2$ we can clear the denominators, turning the above centered equation into

$$x^2+4 = A(x-1)^2 + Bx(x-1) + Cx.$$

Plugging in $x=0$ tells us $A=4$, and plugging in $x=1$ tells us that $C=5$. To get $B$, we can either expand both sides now, or just plug in any other value for $x$; for instance, using $x=2$, we get

$$2^2+4 = 4(2-1)^2 + B(2)(2-1) + 5(2),$$

which when solved tells us $B=-3$. So now,

$$\int \frac{x^2+4}{x(x-1)^2}\,dx = \int \left(\frac{4}{x} + \frac{-3}{x-1} + \frac{5}{(x-1)^2}\right)dx = \int \frac{4}{x}\,dx + \int \frac{-3}{x-1}\,dx + \int \frac{5}{(x-1)^2}\,dx,$$

and we'll win if we can integrate the pieces. The first one is simple:

$$\int \frac{4}{x}\,dx = 4\ln(x).$$

The other two are integrated with the substitution $u = x-1$, $du = dx$. First, we get

$$\int \frac{-3}{x-1}\,dx = \int \frac{-3}{u}\,du = -3\ln(u) = -3\ln(x-1),$$

and for the final piece,

$$\int \frac{5}{(x-1)^2}\,dx = \int \frac{5}{u^2}\,du = 5\,\frac{u^{-1}}{-1} = \frac{-5}{u} = \frac{-5}{x-1}.$$

So, our final answer is

$$\boxed{\int \frac{x^2+4}{x(x-1)^2}\,dx = 4\ln x - 3\ln(x-1) - \frac{5}{x-1} + c.}$$

$\square$

The problem was graded with the partial fractions accounting for eighteen points and the integration itself accounting for seven points. If you wrote down the correct partial fractions decomposition anywhere I awarded eight points. Some people wrote down decompositions that were not exactly what I had on the previous page but would work equally well, for instance

$$\frac{x^2+4}{x(x-1)^2} = \frac{A}{x} + \frac{Bx+C}{(x-1)^2}$$

(which has the solution $A = 4$, $B = -3$, $C = 8$) or even

$$\frac{x^2+4}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{(x-1)^2}.$$

This last one is technically not "as" correct as the other two, because it does not have a unique solution, but there at least *are* solutions, for instance $A = 4$, $B = -6$, $C = 3$, $D = 2$ or $A = 4$, $B = -8$, $C = 5$, $D = 0$, and if you came up with the correct set of constants you could find your way to the correct antiderivative.

Finding the constants was worth ten points; if you were following the solution as I wrote it above, $A$ and $C$ were three points each, $B$ was four points. A number of people tried to clear denominators by multiplying by $x(x-1)(x-1)^2$, which doesn't work (a stray factor $x-1$ doesn't cancel on the left). If your algebra was suspect but you seemed to understand that to find the values of the constants you should equate coefficients or plug values in for $x$, there was some partial credit for those ten points. Equating coefficients involves expanding the right side of the decomposition so that it says

$$x^2 + 4 = (Ax^2 - 2Ax + A) + Bx^2 - Bx + Cx,$$

then deducing from this that the constants $A$, $B$, and $C$ must satisfy the relations $A + B = 1$, $-2A - B + C = 0$, and $A = 4$. This is an equally valid way to find the values of $A$, $B$, and $C$, but in practice it was riskier. I was shocked at how many people failed at multiplying $A(x-1)^2 = A^2 - 2Ax + A$.

The integration itself was worth seven points: two points each for for the two natural logs, and three points for the final integral. I didn't require you to write out the $u$-substitutions explicitly, nor did I care if you think the antiderivative of $\frac{1}{x}$ is $\ln x$ or $\ln|x|$. I did take off one point if you left out the constant of integration. At least **forty people** could not antidifferentiate $\frac{5}{(x-1)^2}$, and if you were one of them, you should be ashamed of yourself!

Also, a number of people who did one of the alternative partial fractions decompositions mentioned above got down to integrating $\frac{Cx+D}{(x-1)^2}$ and then had no idea what to do. The procedure is to apply a $u$-substitution with $u = x - 1$, $du = dx$, and then replace the $x$ in the numerator with $x + 1$. The new numerator is then $Cu + (C + D)$. Then you break the integral up and deal with the two pieces. Here's how it looks:

$$\int \frac{Cx+D}{(x-1)^2}\,dx = \int \frac{C(u+1)+D}{u^2}\,du = \int \frac{Cu}{u^2}\,du + \int \frac{C+D}{u^2}\,du = C\ln(x-1) - \frac{C+D}{x-1} + c.$$