Exam 4 Problem 2:

In order to find the interval of convergence of

$$\sum_{n=1}^{\infty}(-1)^n \frac{\ln(n)x^n}{n},$$

first the radius of convergence is calculated using either the generalized root or ratio test.

Using the ratio test, the power series converges for values of x such that

$$\lim_{n\to\infty} |\frac{\ln(n+1)}{\ln(n)} \cdot \frac{n}{n+1}||x| < 1.$$

We consider the limits of sequences separately, $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$

Then let $f(x) = \frac{\ln(x+1)}{\ln(x)}$ and use L'Hopital's rule and the continuity of f to conclude that

$$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1/(x+1)}{1/x} = \lim_{x\to\infty} \frac{x}{x+1} = 1.$$

Thus we get that the radius of convergence is 1. A similar argument using the root test gives the same answer by calculating $\lim_{n\to\infty}[\ln(n)]^{1/n}$ by taking the limit of the equivalent real-variable function.

This means that there's absolute convergence on the interval $-1 < x < 1$. When $x = 1:$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)(1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

is an alternating series. $\frac{\ln(n)}{n}$ is a decreasing sequence since its equivalent real-variable function $\frac{\ln(x)}{x}$ has a negative derivative $\frac{1-\ln(x)}{x^2}$ is negative for $x > e$. Using a comparison, we obtain

$$0 \leq \lim_{n\to\infty} \frac{\ln(n)}{n} \leq \lim_{n\to\infty} \frac{2\sqrt{n}}{n} \Rightarrow \lim_{n\to\infty} \frac{\ln(n)}{n} = 0.$$

So by the alternating series test, the series converges conditionally at $x = 1$.

When $x = -1:$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n} \ln(n)}{n} = \sum_{n=1}^{\infty} \frac{\ln(n)}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

by the p-series test, hence the series diverges at $x = 1$. This also means that the series does not converge absolutely at $x = -1$.

Therefore the interval of convergence is $(-1, 1]$.

- 10 points for finding the radius of convergence
  - 2 points for using ratio or root test
  - 2 points for using absolute values where necessary
  - 4 points for correctly evaluating limits in ratio or root test computation, only 2 points without correct justification
  - 2 points for getting that the radius is 1
- 7 points for analysis of series at $x = 1$
  - 4 for correct use of alternating series test: 2 points for showing the sequence decreases, 2 points for showing the sequence converges to 0
  - 3 points for concluding that the convergence is conditional, only 1 point if type of convergence is not specified

- 7 points for analysis of series at $x = -1$
  - 5 for for making a valid comparison to a divergent series or correctly using the integral test
  - 2 points for stating that the series diverges
- 1 point for writing the interval $(-1, 1]$