Midterm 3, Problem 3

November 13, 2013

Problem (25pts): Determine whether $\int_0^{\pi} \frac{\cos(x)\mathrm{dx}}{\sqrt{\sin(x)}}$ converges. If so, evaluate.

Solution:

(6pts): Since the integrand is defined on $(0, \pi)$, and

$$\lim_{x \to 0^+} \left(\frac{\cos(x)}{\sqrt{\sin(x)}} \right) = +\infty$$

and

$$\lim_{x \to \pi^-} \left(\frac{\cos(x)}{\sqrt{\sin(x)}} \right) = -\infty,$$

one must break the integral into two improper integrals.

(2pts): So, pick a value $0 < d < \pi$. Then,

$$\int_0^{\pi} \frac{\cos(x)\mathrm{dx}}{\sqrt{\sin(x)}} = \int_0^{d} \frac{\cos(x)\mathrm{dx}}{\sqrt{\sin(x)}} + \int_d^{\pi} \frac{\cos(x)\mathrm{dx}}{\sqrt{\sin(x)}},$$

provided that both of the improper integrals on the right-hand side exist. Note: I will pick $d = \frac{\pi}{2}$. Also OK to leave it as a variable.

(8pts): Then,

$$\begin{aligned} \int_0^{d} \frac{\cos(x)\mathrm{dx}}{\sqrt{\sin(x)}} &= \lim_{a \to 0^+} \int_a^{d} \frac{\cos(x)\mathrm{dx}}{\sqrt{\sin(x)}} \\ &= \lim_{a \to 0^+} \left(2\sqrt{\sin(x)} \Big]_a^d \right) \\ &= 2\sqrt{\sin(d)} \\ &= 2. \end{aligned}$$

(Note: 4 points given for the substitution $u = \sin(x)$, etc, while evaluating the integral; 4 points given for the remainder of the work.)

(8pts): Similarly,

$$\begin{aligned}\int_d^\pi \frac{\cos(x)\mathrm{dx}}{\sqrt{\sin(x)}} &= \lim_{b\to\pi^-} \int_d^b \frac{\cos(x)\mathrm{dx}}{\sqrt{\sin(x)}} \\ &= \lim_{b\to\pi^-} \left(2\sqrt{\sin(x)} \Big]_d^b \right) \\ &= -2\sqrt{\sin(d)} \\ &= -2.\end{aligned}$$

(Note: Points given identically as in the other integral.)

(1pt): Thus,

$$\int_0^\pi \frac{\cos(x)\mathrm{dx}}{\sqrt{\sin(x)}} = 2\sqrt{\sin(d)} + \left(-2\sqrt{\sin(d)}\right) = \boxed{0.}$$