Math 141 Midterm 3 Question 3 Solution

November 10, 2015

Question: Evaluate the following integrals:
a) [10 points]

$$\int \frac{\cos(x)\csc(x)}{\cot^2(x)}\,dx$$

b) [15 points]

$$\int \frac{1}{x^2(x^2+1)}\,dx.$$

Solution to a):

$$\int \frac{\cos(x)\csc(x)}{\cot^2(x)}\,dx$$

$$= \int \frac{\cos(x)\frac{1}{\sin(x)}}{\frac{\cos^2(x)}{sin^2(x)}}\,dx$$

$$= \int \frac{\cos(x)\sin^2(x)}{\cos^2(x)\sin(x)} \qquad \text{3 pts}$$

$$= \int \frac{\sin(x)}{\cos(x)}\,dx \qquad \text{2 pts}$$

Set $u = \cos(x)$ so $du = -\sin(x)\,dx$ and thus

$$= -\int \frac{1}{u}\,du \qquad \text{3 pts}$$

$$= -\ln|\cos(x)| + C \qquad \text{2 pts}$$

Solution to b):
Set up partial fractions:

$$\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} \qquad \text{6 pts}$$

$$\frac{1}{x^2(x^2+1)} = \frac{Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2}{x^2(x^2+1)}$$

$$1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$$

$$1 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

Thus $A + C = 0$; $B + D = 0$; $A = 0$; $B = 1$.

$$A = 0, B = 1, C = 0, D = -1 \qquad \text{4 pts}$$

$$\int \frac{1}{x^2} - \frac{1}{x^2+1}\, dx$$

$$-\frac{1}{x} - \tan^{-1}(x) + C \qquad \text{5 pts}$$

OR we can use trig substitution.
$x = \tan(u)$ so $dx = \sec^2(u)du$ and thus our integral becomes:

$$\int \frac{\sec^2(u)}{\tan^2(u)(\tan^2(u)+1)}\, du \qquad \text{5 pts}$$

$$= \int \cot^2(u)\, du \qquad \text{3 pts}$$

$$= \int \csc^2(u) - 1\, du \qquad \text{4 pts}$$

$$= -\cot(u) - u + C$$

$$= -\cot(\tan^{-1}(x)) - \tan^{-1}(x) + C$$

$$= -\frac{1}{x} - \tan^{-1}(x) + C \qquad \text{3 pts}$$