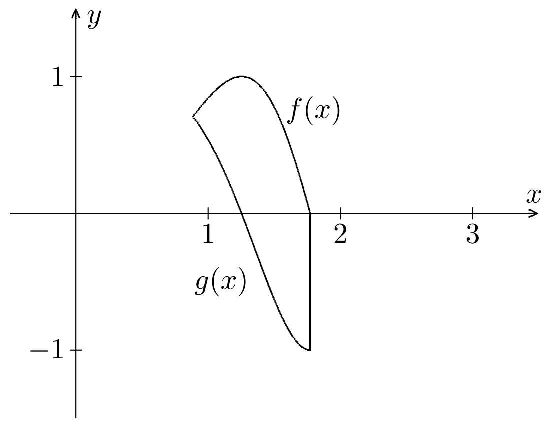
#4. Let R be the region between the graphs of the functions $f(x) = \sin(x^2)$ and $g(x) = \cos(x^2)$ on the interval $\left[\frac{\sqrt{\pi}}{2}, \sqrt{\pi}\right]$. Find the volume of the solid obtained by revolving R about the y-axis. Simplify your answer.

Answer. Here's a picture of the region in question:



(It looks like the graphs of $\sin x$ and $\cos x$, just scrunched a little.) We can see that $f(x) > g(x)$ on the interval in question. Since we're revolving around the y-axis, the volume is calculated with the shell method:

$$\begin{aligned} V &= \int_{\frac{\sqrt{\pi}}{2}}^{\sqrt{\pi}} 2\pi x \left(f(x) - g(x)\right)\, dx \\ &= \int_{\frac{\sqrt{\pi}}{2}}^{\sqrt{\pi}} 2\pi x \left(\sin(x^2) - \cos(x^2)\right)\, dx. \end{aligned}$$

To integrate, perform a u-substitution with $u = x^2$, so $du = 2x\, dx$. Our integrand already had $2x$ in it, so it gets washed away in the substitution. The limits of integration change by $\sqrt{\pi} \mapsto \pi$ and $\frac{\sqrt{\pi}}{2} \mapsto \frac{\pi}{4}$.

$$= \int_{\frac{\pi}{4}}^{\pi} \pi \left(\sin(u) - \cos(u)\right)\, du.$$

Now the antiderivative of $\sin u$ is $-\cos u$ and the antiderviative of $\cos u$ is $\sin u$:

$$\begin{aligned} &= \pi \left(-\cos(u) - \sin(u)\right) \Big|_{\frac{\pi}{4}}^{\pi} \\ &= \pi \left(\left(-\cos(\pi) - \sin(\pi)\right) - \left(-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right)\right). \end{aligned}$$

Finally, the relevant values of sine and cosine are $\cos(\pi) = -1$, $\sin(\pi) = 0$, and $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. We get

$$\begin{aligned} &= \pi \left((- -1 - 0) - (-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}) \right) \\ &= \pi \left(1 + \sqrt{2}\right). \end{aligned}$$

□

The basic rubric was ten points for the setup, ten points for the integration, and five points for the simplification leading to the final answer. The points for integration were split equally between the substitution and the integration of the resulting integrand.