Solution to #4, Test #4

December 5, 2011

#4. Determine whether the following two series converge or diverge.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n^2}$.

Solutions. (a) For the first part, we'll use the integral test on the function $f(x) = \frac{1}{x \ln x}$. For this to be legit, we'll have to check that f is both positive and decreasing. $f(x)$ is positive on the interval $[2, \infty)$ since both x and $\ln x$ are positive over that interval. Easy enough. To see that f is decreasing, I'll take the derivative with the quotient rule:

$$f'(x) = \frac{-(\ln x + x \frac{1}{x})}{(x \ln x)^2} = \frac{-1 - \ln x}{(x \ln x)^2}.$$

The denominator is a square, ergo positive, but the numerator is negative (since, as said above, $\ln x$ is negative for $x \geq 1$). Therefore $f'(x)$ is negative and f is decreasing on $[2, \infty)$.

Now that we've checked the hypotheses, we now know that the series in question has the same convergence/divergence behavior as the improper integral $\int_2^\infty \frac{dx}{x \ln x}$. So let's determine what's up with the integral. The key step is a u-substitution with $u = \ln x$, $du = \frac{1}{x}\, dx$.

$$\begin{aligned}
\int_2^\infty \frac{dx}{x \ln x} &= \lim_{a \to \infty} \int_2^a \frac{dx}{x \ln x} \\
&= \lim_{a \to \infty} \int_{x=2}^{x=a} \frac{du}{u} \\
&= \lim_{a \to \infty} \ln u \Big|_{x=2}^{x=a} \\
&= \lim_{a \to \infty} \ln \ln x \Big|_{x=2}^{x=a} \\
&= \lim_{a \to \infty} (\ln \ln a - \ln \ln 2) = \infty,
\end{aligned}$$

because $\ln x$ goes to infinity with x, and therefore $\ln(\ln x)$ goes to infinity with x. So the integral diverges and therefore the series diverges.

(b) This series is alternating due to the factor $(-1)^{n+1}$, so let's use the alternating series test. There are two things we have to verify: we need the sequence $\{a_n\} = \left\{\frac{n+1}{n^2}\right\}$ to be decreasing, and we need this sequence to have limit zero. The second of these two is the easier to verify: if $g(x) = \frac{x+1}{x^2}$, then we have

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x+1}{x^2} = \lim_{x\to\infty} \frac{1}{2x} = 0;$$

where the first limit $\lim_{n\to\infty} a_n$ is the limit of a sequence, and the other limits are limits of functions, and the next-to-last step is an application of L'Hôpital's Rule. For the claim that the sequence is decreasing, examine the derivative of the function $g(x)$:

$$g'(x) = \frac{x^2 - (2x)(x+1)}{x^4} = \frac{x^2 - 2x^2 - 2x}{x^4} = \frac{-x^2 - 2x}{x^4}.$$

Again, on $[1, \infty)$ the denominator is a square (it is $(x^2)^2$) and therefore positive, and the numerator is visibly negative if x is positive. Therefore the derivative of g is decreasing, so the sequence $\{a_n\}$ is decreasing. Therefore the two conditions of the alternating series test apply; the alternating series test then says that $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges. What do you know, that's the series we cared about. $\square$

These problems were difficult to grade because none of the other tests worked. Especially for part (a), the ratio test and root test both give 1, which is useless. The obvious comparisons, $\frac{1}{n \ln n} < \frac{1}{n}$ and $\frac{1}{n \ln n} < \frac{1}{\ln n}$, both go in the wrong direction, since the "big" series $\sum \frac{1}{n}$ and $\sum \frac{1}{\ln n}$ both diverge. There's no obvious limit comparison either (if you try to compare to $\frac{1}{n}$ for instance, you'll either get 0 or ∞ depending on how you arrange the fraction). If you attempted to do any of those other tests, you probably made no progress—or even worse, you didn't make progress but thought you actually did. It was difficult to impossible to award credit in those cases.

It disturbed me how many people realized that they should use the integral test but didn't check that the integral test was valid. You have to check the hypotheses before you get to use the test! Yes it's work, but math is work! I took off three points if you just said the function was decreasing and offered no explanation and I took off another three points if you didn't discuss the hypotheses at all. If you tried to argue that the function was decreasing without taking the derivative, I was disappointed. This is calculus class, we take derivatives here. The integral was worth about eight and the logic of your argument was worth about six.

For the second part, I took four points if you didn't discuss the hypotheses, so 16 was a common score. If you didn't even say that the series was alternating, I can't tell the difference between "I'm doing the alternating series test and the terms go to zero and that's part of the test" and "I'm going to show the series converges by showing the terms go to zero". Of course this latter strategy is rubbish. If you tried the ratio test, you should have gotten 1, so the test fails. Remember the ratio test only works when the terms of the series are positive, so to apply it you need to apply absolute values. A lot of people forgot the absolute values and got the limit -1. This makes no sense. Other people did a limit comparison, but the limit they used involved $\frac{n+1}{n^2}$. If you do this, you're going to learn that $\sum \frac{n+1}{n^2}$ behaves the same as another series. But you don't know that $\sum \frac{n+1}{n^2}$ behaves the same as $\sum (-1)^{n+1} \frac{n+1}{n^2}$! In fact in this case the two series do **not** behave the same way: the series converges but not absolutely.