

Students who take the course for 3 credits should do all problems. Students who take the course for n credits ($n = 1, 2$) should do problems indicated with n asterisks.

- 1.** (a)(10pts) Prove that the Laplace integral

$$I(x) = \int_0^{\infty} e^{-xt} \phi(t) dt, \quad x > 0,$$

where $\phi(t)$ is infinitely differentiable in $[0, a)$ and $a > 0$, has the asymptotic expansion

$$I(x) \sim \sum_{n=0}^{\infty} \phi^{(n)}(0) x^{-n-1} \quad x \rightarrow +\infty.$$

- (b)(10pts) Find the entire asymptotic expansion as $x \rightarrow +\infty$ of the integral

$$I(x) = \int_0^{\infty} e^{-xt} \frac{t^b}{1+t} dt, \quad x > 0,$$

where b is a real number with $b > -1$. **Hint:** Notice that the statement in (a) does not strictly apply here since the integrand is not infinitely differentiable at $t = 0$ for arbitrary b ($b > -1$). So, you are asked to *modify* (a) a bit, at least heuristically. You may use the Gamma function defined in class.

- 2.* (a)(12pts) Find the entire asymptotic expansion for *large positive* λ of

$$I(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda \sinh^4 t} dt.$$

(b)(8pts) In (a) find a bound for the remainder when the first N terms of the asymptotic expansion are summed. For large positive λ , what is the number of terms, N , to be used for *maximum* accuracy?

- 3.** (20pts) The concentration field $u(x, t)$ of a diffusing dye solves the heat equation,

$$\partial_t u = \nu \partial_{xx} u, \quad \text{with } u(x, 0) = f(x),$$

in one space dimension; $(x, t) \in \mathbb{R} \times (0, +\infty)$ where \mathbb{R} : set of real numbers, the initial data $f(x) \in C^{2N}(\mathbb{R})$ ($f(x) > 0$), and ν is the diffusivity ($\nu > 0$). We also take $f \in L^1(\mathbb{R})$, i.e., $\int_{-\infty}^{\infty} f(x) dx$ is finite. We expect that $u(x, t)$ falls off “sufficiently rapidly” as $|x| \rightarrow \infty$.

(a)(15 pts) Provide the unique solution for u in terms of a single integral; see Lecture 1. Find an asymptotic expansion to N terms for u for “small” t . What is meant by “small t ” here?

(b)(5 pts) Discuss the asymptotic expansion of part (a): Is it convergent or divergent as $N \rightarrow \infty$? (Assuming that $f \in C^\infty(\mathbb{R})$.)