MATH 648M: Advanced Analytic Methods in Applied Mathematics
Department of Mathematics, UMCP **Spring 2009**
Homework 3 **Posted**: Tuesday, 03/03/09
Due: Thursday, 03/26/09

Students who take the course for 3 credits should do all problems (7-9). Students who take the course for n credits ($n = 1$, 2) should do any n problems from problems 7-9.

7. (20 pts) [Exercise on Boundary Layers] Solve approximately the following equations for $0 < \epsilon \ll 1$ by following arguments similar to those given in class. (Solutions that *only* involve applications of theorems *won't be acceptable*):

   (a) $\epsilon u''(x) + (1+x)^2 u'(x) + u(x) = 0,\ u(0) = 0,\ u(1) = 2;\ 0 < x < 1.$

   (b) $\epsilon u''(x) - (1+x^2) u'(x) + u(x) = 0,\ u(0) = 0,\ u(1) = 2;\ 0 < x < 1.$

8. (20 pts) The Reynolds equation in gas lubrication theory is

$$\epsilon \frac{d}{dx}(H^3 u u') = \frac{d}{dx}(Hu), \quad 0 < x < 1,\ 0 < \epsilon \ll 1,$$

   where $u(x)$ is unknown, $\underline{u(0) = u(1) = 1}$, and $H(x)$ is a known, smooth, positive function with $H(0) \neq H(1)$. Find a composite expansion for $u(x)$. **Hint:** Part or all of the solution will be defined implicitly; but it is still possible to match the expansions.

9. (25pts) [Boundary layers for PDE] The following problem arises in the flow of a viscous heat-conducting fluid along a channel. Let $u(x, y)$ satisfy the PDE $\epsilon u_{yy} - (1 - y^2) u_x + 4y^2 = 0$ in the region $x > 0, -1 < y < 1$, where $0 < \epsilon \ll 1; u$ obeys $u(0, y) = 0$ and $u(x, \pm 1) = 1$.

   Describe the solution u by boundary layer theory. Find the outer solution, $u_0(x, y)$. Introduce a boundary layer near $y = -1$ by setting $\zeta = \epsilon^{-\alpha}(1 + y)$. Show that $u_0 = O(\epsilon^{-\alpha})$ near the boundary layer. This suggests writing $u = \epsilon^{-\alpha}\psi(x, \zeta)$ inside the boundary layer. Derive a PDE for ψ and determine α. Give suitable conditions for ψ. **Hint:** You are not asked to solve the derived PDE. What happens near $y = 1$?