MATH 648M: Advanced Analytic Methods in Applied Mathematics
Department of Mathematics, UMCP **Spring 2009**
Homework 6 **Posted**: Wednesday, 04/22/09
Due: Thursday, 05/07/09

Students who take the course for 1 credit should do any 2 problems from Probs. 16-19. Students who need 3 credits should do all problems.

16. (20 pts) [More on stochastic equations.] Particles moving on a plane have coordinates $(x_1(t), x_2(t))$ that satisfy the *pair* of stochastic equations

$$dx_1 = a_1\, dt + dw_1\ , \quad dx_2 = a_2\, dt + dw_2\ ,$$

where a_1, a_2 are constants and w_1, w_2 are independent BM's. Find the PDE ("Fokker-Planck equation") satisfied by the joint probability density function $W(x, y, t)$ of the random variables x_1 and x_2.

17. (20 pts) [Wiener measure] Evaluate exactly the mean $E(F[w]) = \int F\, dW$ (where $\int dW$: Wiener measure) for the following functionals:

(i) (10 pts) $F[w(\cdot)] = \int_0^1 w(s)^4\, ds$;

(ii) (10 pts) $F[w(\cdot)] = \sin(w(1)^3)$.

18. (20 pts) Explain which ones of the following functions can be the *covariance* functions of some stationary stochastic process ($T = t_2 - t_1$ as usual). Justify your answer.

(i) (5 pts) $R(T) = e^{-T^2}$; (ii) (5 pts) $R(T) = Te^{-T^2}$.

(iii) (5 pts) $R(T) = (T^2 - 1)e^{-T^2/2}$; (iv) (5 pts) $R(T) = (1 - T^2)e^{-T^2/2}$.

19. (24pts) [Modeling of fluctuations of defects on crystal surfaces.] The problem of fluctuations of line defects ("steps") on crystal surfaces is sometimes mapped to a 1D system of N particles interacting on a line. The particle positions $x_m(t)$ obey the coupled stochastic differential equations (SDEs)

$$dx_m = -\gamma x_m\, dt + \left(\sum_{\substack{l=1 \\ l \neq m}}^{N} \frac{\rho}{x_m - x_l} \right) dt + dw\ , \quad m = 1, \ldots, N,$$

where "dw/dt" is white noise, $\gamma > 0$, and the constant ρ is the effective "charge" of each particle. Of interest is the statistics of the next-neighbor distances ("gaps") $g_m := x_{m+1} - x_m$.

(a) (5 pts) Find the coupled SDEs for $g_m(t)$. **Hint:** These equations will involve products of distances between particles in denominators inside sums.

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(b) (7 pts) Simplify the equations derived in part (a) by replacing products of form $(x_k - x_{m+1})(x_k - x_m)$ in denominators inside sums by their means, where the averages are taken in the stationary state; i.e., apply

$$(x_k - x_{m+1})(x_k - x_m) \approx \langle g^2 \rangle_{st}(k - m - 1)(k - m),$$

where $\langle g^2 \rangle_{st}$ denotes the mean of g_m^2 (same for every m) at the stationary state; treat this parameter as known. The above approximation amounts to a "mean-field" approach. **Hint:** You should arrive at a one-particle (decoupled) SDE. You may use $\sum_{p=1}^{\infty}[p(p+1)]^{-1} = 1$.

(c) (12 pts) Find the desired PDE for the pdf of $g = g_m$ from the simplified SDE derived in part (b), by using the scaled variables $s = g/\langle g \rangle_{st}$ and $\tilde{t} = t/\langle g^2 \rangle_{st}$. What is the equation for the pdf of g as $t \to +\infty$? Discuss that the pdf for long times approaches the Wigner form $As^{\rho}e^{-bs^2}$ where $A > 0$ and $b > 0$ depend on ρ. **Remark:** This distribution is the celebrated "Wigner surmise". **Hint:** For part (c) of this problem, you may want to use the result of Prob. 15(b), Homework 5.