MATH 673 (AMSC 673): Partial Differential Equations I
Department of Mathematics, UMCP Fall 2010

Midterm Exam

Handed out: Wednesday, 10/20/10

Do ALL Problems 1-3. Problems 1-3 are equivalent. Explain your steps carefully. If you invoke a “well known” theorem from the textbook, make clear precisely which theorem you are using and justify its use.

1. Let \( u(x) \) be a harmonic function in \( \mathbb{R}^n, \ n \geq 2 \). Suppose that \( \int_{\mathbb{R}^n} |Du(x)|^p \, dx < \infty \).

   (a) For \( p = 2 \), prove or disprove that \( u \) is a constant.

   (b) Now consider the more general case \( p > 1 \). Prove or disprove that \( u \) is a constant.

2. Suppose that the function \( u : U \times [0,T] \rightarrow \mathbb{R} \ (U \subset \mathbb{R}^n) \), where \( u \in C^2(U_T) \cap C(\overline{U_T}) \), is a supersolution of the heat equation, i.e.,

\[
  u_t - \Delta u \geq 0 \quad \text{in} \quad U_T := U \times (0,T);
\]

here, \( U \) is bounded and open and \( \partial U \) is compact. Show that

\[
  \min_{\overline{U_T}} u = \min_{\Gamma_T} u
\]

where \( \Gamma_T := \overline{U_T} - U_T \) is the parabolic boundary of \( U_T \).

3. Suppose that \( u \) is smooth and solves the heat equation, \( u_t - \Delta u = 0 \) in \( \mathbb{R}^n \times (0,\infty) \).

   (a) Show that \( u_{\lambda} = u(\lambda x, \lambda^2 t) \) solves the heat equation for every \( \lambda \in \mathbb{R} \).

   (b) Use part 3(a) to show that \( v(x,t) := x \cdot (D^2 u) x + 2t(2x \cdot Du_t + u_t + 2tu_{tt}) \) solves the heat equation.

**Note:** For part 3(b) you will receive only partial credit if you do not use part 3(a). Recall that if \( A = ((a_{ij})) \) is an \( n \times n \) matrix, then \( x \cdot Ax = \sum_{i,j=1}^n a_{ij} x_i x_j \) is the corresponding quadratic form.