1 Volume:

$$V = \iint_R (4-x^2-y^2)\, dA \quad , \quad R = \{ |x| \le 1,\ |y| \le 1 \} : \text{square} \qquad [3pts]$$

$$= \int_{-1}^{1}\int_{-1}^{1} (4-x^2-y^2)\, dy\, dx \qquad [2pts]$$

$$= \int_{-1}^{1} \left((4-x^2)y - \frac{y^3}{3}\right)\Bigg|_{y=-1}^{1} dx = \int_{-1}^{1} \left[2\,(4-x^2) - \frac{2}{3}\right] dx = \int_{-1}^{1} \left(\frac{22}{3} - 2x^2\right) dx \qquad [3pts]$$

$$= \left(\frac{22}{3}x - 2\,\frac{x^3}{3}\right)\Bigg|_{-1}^{1} = \frac{22}{3}\cdot 2 - 2\cdot\frac{2}{3} = \frac{40}{3} \qquad [2pts]$$

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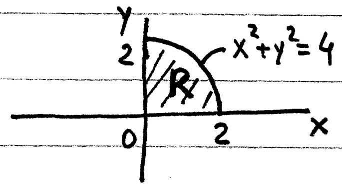
$$I = \int_0^2 \int_0^{\sqrt{4-y^2}} e^{x^2+y^2}\, dx\, dy = \iint_R e^{x^2+y^2}\, dA$$



where R is the portion of circular disk $x^2+y^2 \le 4$ in the first quadrant. [3pts]

[TA's : The students do not need to sketch R but must describe R adequately.]

Use polar coordinates : $x = r\cos\theta$, $y = r\sin\theta$; $0 \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$. [2pts]

$$I = \int_0^{\pi/2}\int_0^2 e^{r^2}\, r\, dr\, d\theta \qquad [3pts]$$

$$= \int_0^{\pi/2} \left(\frac{1}{2}e^{r^2}\right)\Bigg|_0^2 d\theta = \frac{1}{2}\,(e^4-1)\cdot \frac{\pi}{2} = \frac{\pi}{4}(e^4-1) \qquad [2pts]$$

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3 Total mass $m = \iiint_D \delta(x,y,z)\, dV = \iiint_D z\, dV$ ($|z| = z$ above x-y plane) [2pts]

where D is the solid region above xy plane between spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$.

Use spherical coordinates: (ρ, φ, θ).

In the present problem, $1 \le \rho \le 3$, $0 \le \varphi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$.

$$\Rightarrow I = \int_0^{2\pi} \int_0^{\pi/2} \int_1^3 \underbrace{(\rho\cos\varphi)}_{z} \overbrace{\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}^{dV} \qquad [3pts]$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \left(\frac{\rho^4}{4}\right)\Bigg|_1^3 \sin\varphi\cos\varphi \, d\varphi \, d\theta = 20 \int_0^{2\pi} \int_0^{\pi/2} \sin\varphi \cos\varphi \, d\varphi \, d\theta \qquad [2pts]$$

$$= 20 \int_0^{2\pi} \left(\int_0^{\pi/2} \sin\varphi \, d(\sin\varphi) \right) d\theta = 20 \int_0^{2\pi} \left(\frac{\sin^2\varphi}{2}\right)\Bigg|_0^{\pi/2} d\theta \qquad [2pts]$$

$$= 20 \cdot \frac{1}{2} \sin^2\left(\frac{\pi}{2}\right) \cdot 2\pi = 20\pi \qquad [1pt]$$

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[4] Let $u = x - 2y$, $v = x + 2y$. Then, $x = \frac{u+v}{2}$, $y = \frac{v-u}{4}$ [2pts]

This is a convenient change of variables:

- The line $x - 2y = c$ maps to $u = c$ $(c = 1, 2)$
- The line $x + 2y = c$ maps to $v = c$ $(c = 1, 3)$

Thus, R is transformed to $S = \{ 1 \le u \le 2,\ 1 \le v \le 3 \}$, a rectangle. [2pts]

Jacobian: $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/4 & 1/4 \end{vmatrix} = \frac{1}{4} \qquad [2pts]$$

$$I = \iint_S \frac{u^2}{v^2} \frac{1}{4} \, dA$$ [TA's: Students should define S; no sketch needed] [1pt]

$$= \frac{1}{4} \int_1^2 \int_1^3 \frac{u^2}{v^2} \, dv \, du = \frac{1}{4} \int_1^2 u^2 \left(\frac{-1}{v}\right)\Bigg|_1^3 du = \frac{1}{4} \cdot \frac{2}{3} \int_1^2 u^2 du \qquad [2pts]$$

$$= \frac{1}{6} \left(\frac{u^3}{3}\right)\Bigg|_1^2 = \frac{1}{6} \cdot \frac{2^3 - 1}{3} = \frac{7}{18} \qquad [1pt]$$