MATH 445 — Exam 1 Review

Topics

• sentences, truth assignments, and satisfaction
• validity, satisfiability, and the relation between them
• truth tables and other ways of determining validity and satisfiability
• logical consequence and its relation to satisfiability
• ways of determining logical consequence
• equivalence, rewriting, DNF, CNF
• logical axioms, MP, deductions, and Soundness
• deductions from hypotheses, the Deduction Theorem, Soundness
• consistent sets of sentences: definition and characterization
• the relation between deducibility and consistency
• extending consistent sets, maximal consistent sets
• consistent sets are satisfiable, Completeness Theorem
• Compactness Theorem

Sample Problems

1. Assume that $h \models (A \lor B) \rightarrow (A \land B)$. What can you conclude about $h(A)$ and $h(B)$? Explain carefully.

2. Write down the complete truth tables for each of the following:

   (a) $((A \rightarrow B) \rightarrow B) \rightarrow B$

   (b) $((A \rightarrow B) \lor (B \rightarrow C)) \rightarrow (B \lor C)$

3. Use the equivalences given in the notes (not truth tables) to find the DNF and CNF of $(A \rightarrow \neg C) \rightarrow (B \rightarrow \neg C)$. 

1
4. Determine which of the following are valid and which are satisfiable but not valid. Explain – in particular for those which are satisfiable but not valid you must give both an assignment satisfying the sentence and an assignment falsifying it.

(a) \((A \rightarrow B) \rightarrow C) \rightarrow (A \rightarrow C)\)
(b) \((A \rightarrow B) \rightarrow C) \rightarrow (\neg A \rightarrow C)\)
(c) \((A \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow C)\)
(d) \((\neg A \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow C)\)

5. Prove or disprove (with a counterexample) each of the following:

(a) If both \(\varphi\) and \(\psi\) are satisfiable then so is \(\varphi \land \psi\).
(b) If \(\varphi \models \psi\) then \(\models (\varphi \rightarrow \psi)\).
(c) If \(\models (\varphi \rightarrow \psi)\) then \(\psi\) is satisfiable.
(d) If \(\varphi \models \theta\) and \(\psi \models \theta\) then \(\varphi \lor \psi \models \theta\).

6. Establish the following. You may use the Deduction Theorem and the facts that \(\vdash \varphi \rightarrow \varphi\), \(\vdash (\neg \varphi \rightarrow \varphi) \rightarrow \varphi\), and \(\vdash \varphi \rightarrow (\neg \varphi \rightarrow \psi)\) but no other derived facts.

(a) \(\vdash ((\varphi \rightarrow \psi) \rightarrow (\psi \rightarrow \theta)) \rightarrow (\psi \rightarrow \theta)\)
(b) \(\vdash (\neg \varphi \rightarrow \psi) \rightarrow (\neg \psi \rightarrow \varphi)\)

7. (a) Define what it means for a set \(\Sigma\) to be consistent.

(b) Prove that if \(\Sigma\) is inconsistent then \(\Sigma \vdash \psi\) for every sentence \(\psi\).

(c) Define what it means for \(\Sigma\) to be maximal consistent.

8. Prove or disprove (with a counterexample) each of the following:

(a) If \(\Sigma \cup \{\varphi\}\) is inconsistent then \(\Sigma \cup \{\neg \varphi\}\) is consistent.
(b) If \(\Sigma \cup \{\varphi\}\) is consistent then \(\Sigma \cup \{\neg \varphi\}\) is inconsistent.

9. (a) State the Completeness Theorem.

(b) State the Compactness Theorem.

(c) Prove the Compactness Theorem, using the Completeness Theorem.