1. [5 pts] Let $\mathcal{L}$ be the language whose non-logical symbols are a unary function symbol $F$ and binary function symbols $G$ and $H$. Let $\mathcal{A}$ be the $\mathcal{L}$-structure with universe $\mathbb{Z}$ which interprets $F$ by $p(x)$ (where $p(k) = k - 1$), interprets $G$ by $+$, and interprets $H$ by $-$, and let $\mathcal{B}$ be the $\mathcal{L}$-structure with universe $\mathbb{Z}$ which interprets $F$ by $s(x)$ (where $s(k) = k + 1$), interprets $G$ by $\cdot$, and interprets $H$ by $-$. Determine $t^{\mathcal{A}}$ and $t^{\mathcal{B}}$ where

$$t = H(G(F(x), F(y)), F(G(x, y)))$$

Simplify your answers.

2. Let $\varphi$ be the formula

$$\exists y R(x, y) \land \exists z R(y, z) \rightarrow \forall x (R(x, z) \rightarrow \exists z R(z, x))$$

[2 pts] (a) List all subformulas of $\varphi$.

[2 pts] (b) Determine which occurrences of variables in $\varphi$ are free and which are bound.

3. [5 pts] Let $\mathcal{L}$ be the language whose only non-logical symbol is a binary relation symbol $R$. Let $\mathcal{A} = (\mathbb{N}, \leq)$. Determine the sets defined by the following formulas in $\mathcal{A}$. Explain and simplify your answers.

(a) $\varphi(x)$ is $\forall y (R(y, x) \rightarrow R(x, y))$

(b) $\psi(x)$ is $\forall y (R(x, y) \rightarrow R(y, x))$

4. [6 pts] Let $\mathcal{L}$ be the language whose only non-logical symbols are unary relation symbols $E$ and $P$ and the binary relation symbol $R$. Let $\mathcal{A}$ be the $\mathcal{L}$-structure with universe $\mathbb{N}$ which interprets $E$ by the set of even natural numbers, $P$ by the set of prime natural numbers, and $R$ by $<$. Give sentences of $\mathcal{L}$ which “naturally” express the following facts about $\mathcal{A}$.

(a) There is a smallest prime number.

(b) There is no largest prime number.

(c) There is exactly one even prime number.

NOTE: Your solutions must include enough detail to justify your conclusions.