CHAPTER 3. THE COMPLETENESS THEOREM

1. INTRODUCTION

In this Chapter we prove Gödel’s Completeness Theorem for first order logic.

Theorem 1.1. (Completeness Theorem) Let $\Sigma \subseteq S_{nL}$.

(a) $\Sigma$ is satisfiable iff $\Sigma$ is consistent.

(b) For any $\theta \in S_{nL}$, $\Sigma \vdash \theta$ iff $\Sigma \models \theta$.

By Soundness (Theorem 8.2 in Chapter 2) we know the left to right implications of both (a) and (b). Recall also Lemma 6.2 and Theorem 9.4 from Chapter 2 which assert that $\Sigma \models \theta$ iff $(\Sigma \cup \{\neg \theta\})$ is not satisfiable and that $\Sigma \vdash \theta$ iff $(\Sigma \cup \{\neg \theta\})$ is not consistent. Therefore part (b) of Theorem 1.1 follows from part (a). So it suffices the establish the following result.

Theorem 1.2. (Model Existence) Let $\Sigma \subseteq S_{nL}$ be consistent. Then $\Sigma$ is satisfiable, (i.e., $\Sigma$ has a model).

As in sentential logic the argument for Theorem 1.2 will involve maximal consistent sets of sentences (see below), but we will have to expand the original consistent set by “adding witnesses”, a novel and important technique introduced by Leon Henkin in 1949.

The following definition is verbally the same as in sentential logic.

Definition 1.1. A set $\Gamma \subseteq S_{nL}$ is maximal consistent iff it is consistent and for every $\theta \in S_{nL}$ either $\theta \in \Gamma$ or $\neg \theta \in \Gamma$.

The lemma allowing us to extend consistent sets to maximal consistent sets is stated and proved exactly as in sentential logic.

Lemma 1.1. Let $\Sigma \subseteq S_{nL}$ be consistent and let $\theta \in S_{nL}$. Then either $(\Sigma \cup \{\theta\})$ or $(\Sigma \cup \{\neg \theta\})$ is consistent.

We next note that Finiteness is established exactly as for sentential logic.

Theorem 1.3. Let $\Sigma \subseteq S_{nL}$.

(a) For any $\theta \in S_{nL}$, $\Sigma \vdash \theta$ iff there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash \theta$.

(b) $\Sigma$ is consistent iff every finite $\Sigma_0 \subseteq \sigma$ is consistent.

Finally the result on the existence of maximal consistent sets, due to Lindenbaum, is also stated and proved just as in sentential logic.

Theorem 1.4. Let $\Sigma \subseteq S_{nL}$ be consistent. Then there is some maximal consistent $\Gamma \subseteq S_{nL}$ such that $\Sigma \subseteq \Gamma$.
Note that $\Gamma$ is not usually uniquely determined by $\Sigma$. For example, if both $\Sigma \cup \{\theta\}$ and $\Sigma \cup \{\neg \theta\}$ are consistent, then there will be maximal consistent sets $\Gamma_1$ containing $\Sigma \cup \{\theta\}$ and $\Gamma_2$ containing $\Sigma \cup \{\neg \theta\}$.

2. Adding “Witnesses”

In sentential logic we could define a (unique) truth assignment from a maximal consistent set $\Gamma$ and then show that it satisfied $\Gamma$. In first order logic, however, a maximal consistent set does not determine any structure – to determine a structure we need to know what the universe of the structure is and how the non-logical symbols of the language are interpreted on this set.

Given a consistent set $\Sigma \subseteq Sn_L$, we first expand the language $L$ to $L'$ by adding constant symbols $c_i$ and find a maximal consistent set $\Gamma \subseteq Sl_{L'}$ which also has the property that for every formula $\psi(x)$ of $L'$, $\forall x \psi(x) \in \Gamma$ iff $\psi(c_i) \in \Gamma$ for every $i$. We will then be able to define an $L'$-structure $A'$ such that every element of the universe is named by some constant symbol and which satisfies $\Gamma$.

To simplify the presentation we assume that the only non-logical symbol of the language $L$ is a binary relation symbol $R$. We start with a consistent set $\Sigma \subseteq Sn_L$. We define the language $L' = L \cup \{c_i : i \in \mathbb{N}\}$. The consistency of $\Sigma$ is not affected by this change since the set $\Sigma$ says nothing about the added constants.

We next list all formulas of $L'$ with just one free variable as

$\psi_1(x), \psi_2(x), \ldots, \psi_n(x), \ldots$ for all $n \in \mathbb{N}$.

(The formulas need not have the same free variable; we use $x$ for whatever the free variable in question is). We define a sequence of sets of sentences of $L'$, beginning with $\Sigma$, as follows:

- let $c_1$ be the first constant not appearing in $\psi_1(x)$, let $\theta_1$ be $(\exists x \psi_1(x) \rightarrow \psi_1(c_1))$, and let $\Sigma_1 = \Sigma \cup \{\theta_1\}$. We claim that $\Sigma_1$ is consistent. Otherwise, $\Sigma \vdash \neg \theta_1$, hence $\Sigma \vdash \exists x \psi_1(x)$ and $\Sigma \vdash \neg \psi_1(c_1)$. But then, by Generalization on Constants (Theorem 9.3 in Chapter 2), we would have $\Sigma \vdash \forall x \neg \psi_1(x)$, and so $\Sigma$ is inconsistent (remembering that $\exists x$ means $\neg \forall x \neg$).

We continue in this way, at the $n$th stage adding some sentence $\theta_n$ of the form $\exists x \psi_n(x) \rightarrow \psi_n(c_{i_n})$ to obtain a consistent set $\Sigma_n$. By Theorem 1.3 the union of all of these sets is a consistent set $\Sigma'$.

Now we apply Theorem 1.4 to obtain a maximal consistent $\Gamma \subseteq Sn_{L'}$ such that $\Sigma' \subseteq \Gamma$.

We claim that $\Gamma$ has the following properties:

(i) for every $\theta \in Sn_L$, $\neg \theta \in \Gamma$ iff $\theta \notin \Gamma$,

(ii) for every $\varphi, \theta \in Sn_L$, $(\varphi \rightarrow \theta) \in \Gamma$ iff either $\neg \varphi \in \Gamma$ or $\theta \in \Gamma$,

(iii) for every $\psi(x) \in Fm_L$, $\forall x \psi(x) \in \Gamma$ iff $\psi(c_n) \in \Gamma$ for every $n \in \mathbb{N}$.

This claim is easily proved using the following Lemma.

**Lemma 2.1.** Let $\Gamma$ be maximal consistent. Then for any sentence $\theta$, $\theta \in \Gamma$ iff $\Gamma \vdash \theta$. 
3. Defining a Structure from $\Gamma$

So, given $\Gamma$ obtained as in the preceding section, we define an $\mathcal{L}'$-structure $\mathcal{A}'$ from $\Gamma$ as follows:

- the universe $A'$ of the structure is $\mathbb{N}$;
- $c_n A' = n$ for every $n \in \mathbb{N}$;
- $R^A'(k, n)$ holds iff $R(c_k, c_n) \in \Gamma$.

We obtain the following result.

**Theorem 3.1.** Let $\Gamma$ and $\mathcal{A}'$ be as above. Then for every sentence $\theta$ of $\mathcal{L}'$ in which $=$ does not occur, $\mathcal{A}' \models \theta$ iff $\theta \in \Gamma$.

**Proof.** (outline) We prove this by induction. The base case is $R(c_k, c_n)$, which holds due to the definition of $R^\mathcal{A}'$. The inductive steps for the connectives are clear from parts (i) and (ii) of the Claim at the end of the preceding section. The inductive step for $\forall$ is clear from part (iii) of the Claim and the fact that $A' = \mathbb{N} = \{c_n A' : n \in \mathbb{N}\}$. \hfill $\Box$

To define a structure which will also model the sentences in $\Gamma$ which contain $=$ we need to allow for the possibility that $c_k = c_n \in \Gamma$ for some $k \neq n$. We choose the logical axioms for $=$ to guarantee that $\Gamma$ has the following additional properties:

(iv) if $k \in \Gamma$, if $c_k = c_n \in \Gamma$ then $c_n = c_k \in \Gamma$, and if $c_k = c_n, c_n = c_m \in \Gamma$ then $c_k = c_m \in \Gamma$;

(v) if $R(c_k, c_n), c_k = c_l$, and $c_n = c_m \in \Gamma$ then $R(c_l, c_m) \in \Gamma$.

We define a structure $\mathcal{B}'$ as follows:

- the universe $B'$ of $\mathcal{B}'$ is $\{k \in \mathbb{N} : c_k \neq c_l \in \Gamma$ for all $l < k\}$;
- $c_n B'$ is the least $k$ such that $(c_k = c_n) \in \Gamma$;
- $R^\mathcal{B}'(k, l)$ holds iff $R(c_k, c_l) \in \Gamma$.

We then have the following, proved like Theorem 3.1 using the additional properties (iv) and (v) to check that equality statements in $\Gamma$ are true in $\mathcal{B}'$.

**Theorem 3.2.** For every $\theta \in S_{\mathcal{L}'}$, $\mathcal{B}' \models \theta$ iff $\theta \in \Gamma$.

Since $\Sigma \subseteq \Gamma$ we have shown that the original $\Sigma$ has a model, establishing Theorem 1.2.

**Example.** We illustrate the Henkin method with a simplified example. Let $\Sigma = \{\forall y \exists x R(y, x)\}$. Instead of listing all formulas $\psi(x)$ of $\mathcal{L}'$ we consider only the formulas $\psi_n(x)$ defined as $R(c_n, x)$. Then $\theta_1$ is $(\exists x R(c_1, x) \rightarrow R(c_1, c_2))$, and in general $\theta_n$ is $(\exists x R(c_n, x) \rightarrow R(c_n, c_{n+1}))$, so $\Sigma'$ is

$$\{\forall y \exists x R(x, y) \uplus \{(\exists x R(c_n, x) \rightarrow R(c_n, c_{n+1})) : n \in \mathbb{N}\}$$.

Now let $\Gamma \subseteq S_{\mathcal{L}'}$ be maximal consistent and contain $\Sigma'$. Then $\Gamma \vdash \exists x R(c_n, x)$ for every $n \in \mathbb{N}$ (since $\forall y \exists x R(y, x) \rightarrow \exists x R(c_n, x)$) is an instance of Axiom 2). Therefore $\Gamma \vdash R(c_n, c_{n+1})$, and so $R(c_n, c_{n+1}) \in \Gamma$ by Lemma 2.1, for every $n \in \mathbb{N}$. Thus $\mathcal{A}' \models \forall y \exists x R(y, x)$, as desired, since $A' = \mathbb{N}$. 


4. The Compactness Theorem

Just as in Sentential Logic, the Compactness Theorem is an immediate consequence of Finiteness and the Completeness Theorem.

**Theorem 4.1.** (Compactness Theorem) Let $\Sigma$ be a set of sentences of $L$.

(a) If $\Sigma \models \varphi$ then there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \varphi$.

(b) If every finite $\Sigma_0 \subseteq \Sigma$ has a model then $\Sigma$ has a model.

This result has many amazing applications and is of fundamental importance to the research area of Model Theory. We give two examples.

**Theorem 4.2.** Let $\theta \in S\varphi_L$. Assume that $A \models \theta$ for every $A$ with an infinite universe $A$. Then there is some integer $n$ such that $A \models \theta$ for every $A$ whose universe $A$ contains at least $n$ elements.

*Proof.* For every $k \in \mathbb{N}$ there is a sentence $\sigma_k$ which holds of a structure iff the universe of the structure contains at least $k$ elements. Let $\Sigma = \{\sigma_k : k \in \mathbb{N}\}$. Then $A \models \Sigma$ iff the universe of $A$ is infinite. Therefore the hypothesis of the Theorem implies that $\Sigma \models \theta$. By Compactness there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \theta$. There must be a largest integer $n$ such that $\sigma_n \in \Sigma_0$, and this $n$ is then as desired. $\square$

**Theorem 4.3.** Let $L$ be the language whose only non-logical symbol is a binary relation symbol $\prec$. Let $A$ be the $L$-structure $(\mathbb{N}, \prec)$. Then there is some $B = (B, \prec_B)$ such that $B \models \theta$ for every $\theta$ true on $A$, but $B$ contains “infinite” elements, that is elements $b$ with infinitely many elements in $B$ preceding it in the order $\prec_B$.

*Proof.* For every $n \in \mathbb{N}$ there is a formula $\varphi_n(x)$ of $L$ which holds of an element iff there are at least $n$ elements preceding it. Let $c$ be a constant symbol and let $L' = L \cup \{c\}$. We define the set $\Sigma$ of $L'$-sentences as

$$\{\theta \in S\varphi_L : A \models \theta\} \cup \{\varphi_n(c) : n \in \mathbb{N}\}.$$  

If $B' \models \Sigma$ then $B = (B', \prec_B')$ is as desired, since $c^{B'}$ is an “infinite” element of $B'$. Every finite $\Sigma_0 \subseteq \Sigma$ has a model $A' = (\mathbb{N}, <, n_0)$ where $n_0$ is a sufficiently large element of $\mathbb{N}$, so $\Sigma$ has a model by Compactness. $\square$