1. [10 pts] (a) Let $\mathcal{L}^{nl} = \{R, E, O\}$ where $R$ is a binary relation symbol and $E, O$ are unary relation symbols. Let $\mathcal{A}$ be the $\mathcal{L}$-structure with universe the set of positive integers and which interprets $R$ by $|$ (divides), $E$ by the set of even positive integers, and $O$ by the set of odd positive integers. Give sentences of $\mathcal{L}$ which “naturally” express the following facts about $\mathcal{A}$.

(i) Every odd number divides some even number.

(ii) Some even numbers have no odd divisors.

15 pts] (b) Let $\mathcal{L}^{nl} = \{F\}$ where $F$ is a binary function symbol. Let $\mathcal{A}$ be the $\mathcal{L}$-structure $\langle \mathbb{Z}, + \rangle$ and let $\mathcal{B}$ be the $\mathcal{L}$-structure $\langle \mathbb{Q}, + \rangle$. Give a sentence $\sigma$ of $\mathcal{L}$ such that $\mathcal{A} \models \sigma$ but $\mathcal{B} \not\models \sigma$. Explain.

2. Working directly from the definitions, either prove or find a counterexample to each of the following. Explain carefully.

[15 pts] (a) $\models \exists x(P(x) \rightarrow Q(x)) \rightarrow (\exists xP(x) \rightarrow \exists xQ(x))$

[15 pts] (b) $\models (\forall xP(x) \rightarrow \exists xQ(x)) \rightarrow \exists x(P(x) \rightarrow Q(x))$

3. [15 pts] Using the rules in the notes (but not, of course, the Completeness Theorem) show that

$\models \forall x(P(x) \rightarrow Q(x)) \rightarrow (\exists xP(x) \rightarrow \exists xQ(x))$

4. [15 pts] Either prove (directly from the definitions) or disprove (with a counterexample) each of the following:

(a) If $\exists x \varphi(x)$ and $\exists x \psi(x)$ are both valid then so is $\exists x (\varphi(x) \land \psi(x))$.

(b) $\forall x \varphi(x)$ is valid iff $\exists x \neg \varphi(x)$ is not satisfiable.

5. [5 pts] (a) Define what it means for a set $\Sigma$ of sentences of $\mathcal{L}$ to be maximal consistent.

[10 pts] (b) Let $\Sigma$ be a maximal consistent set of sentences of $\mathcal{L}$. Assume that $\forall x P(x) \in \Sigma$. Prove that $P(c) \in \Sigma$ for every constant $c \in \mathcal{L}$. (Do not assume the Completeness Theorem).

NOTE: Your solutions must include enough detail to justify your conclusions.