1. [10 pts] (a) Explicitly define a linear order $\leq^*$ on $\mathbb{N}$ which is not a well order, but is such that $\mathbb{N}$ has a smallest element under $\leq^*$. Justify your answer.

[15 pts] (b) Explicitly define a well order $\leq^\#$ on $\mathbb{N}$ which has exactly two limit points. Justify your answer. Remember that $0_U$ is not a limit point of a well ordered set.

2. [15 pts] (a) Prove that every order-preserving injection $\pi$ of a well ordered set $U$ into itself is expansive, that is, satisfies $x \leq \pi(x)$ for all $x \in U$.

[10 pts] (b) Use the result in (a) to prove that no well ordered set is similar to a proper initial segment of itself.

3. [20 pts] Assume that for every set $C$ of non-empty pairwise disjoint sets there is a function $h : C \to \bigcup C$ such that $h(X) \in X$ for all $X \in C$. Prove directly that the full Axiom of Choice holds.

4. (AC) Given indexed families $\{\kappa_i\}_{i \in I}$ and $\{\lambda_i\}_{i \in I}$ of cardinal numbers such that $\kappa_i \leq \lambda_i$ for all $i \in I$, establish the following. You may use the facts established in class about finite cardinal sums and products, but only the definition of the infinite sum.

[15 pts] (a) $\sum_{i \in I} \kappa_i \leq c \sum_{i \in I} \lambda_i$.

[5 pts] Give an example to show we cannot conclude $< c$ in (a). Justify your answer.

[10 pts] Assume that $\kappa_i = c \kappa$ for all $i \in I$, where $\kappa$ is some infinite cardinal, prove that $\sum_{i \in I} \kappa_i = c \kappa \cdot |I|$.

NOTE: Your solutions must include enough detail to justify your conclusions.