MATH 446 – Homework 8
(due Monday 30 April 2012)

Note: $\alpha$, $\beta$, and $\gamma$ are all ordinals.

1. [6 pts] Prove, using only the Zermelo Axioms, that the Axiom of Replacement is equivalent to the following:
   for every definite operation $F$ and every set $A$ there is some set $B$ such that $A \subseteq B$ and $F[B] \subseteq B$.

2. [6 pts] Prove that $\alpha + \beta < \alpha + \gamma$ iff $\beta < \gamma$.

3. [6 pts] (a) Prove that for every $\gamma \geq \alpha$ there is exactly one $\beta$ such that $\gamma = \alpha + \beta$.
   [3 pts] (b) Give examples (with proof) of ordinals $\alpha_0$, $\alpha_1$, and $\beta$ such that $\alpha_0 + \beta = \alpha_1 + \beta$ but $\alpha_0 \neq \alpha_1$.

4. A definite operation $F : ON \to ON$ is normal if $\alpha < \beta$ implies $F(\alpha) < F(\beta)$ and for every limit ordinal $\xi$, $F(\xi) = \bigcup \{ F(\alpha) \mid \alpha < \xi \}$.
   [6 pts] Prove that for every normal operation $F$ the following hold:
   $\alpha \leq F(\alpha)$ for all $\alpha$ and $F(\xi)$ is a limit ordinal for every limit ordinal $\xi$.
   [3 pts] Assume that $F$ and $G$ are both normal. Prove that $H(\alpha) = F(G(\alpha))$ is also normal.

NOTE: Your solutions must include enough detail to justify your conclusions.