AMSC 466 - Final - Spring 2016 - Solutions

1. Making the Integral exact for

$$f(x)=x,\quad f(x)=x^3,\quad f(x)=\cos\frac{\pi x}{2} \implies \begin{cases} A_1+2A_2=0 \\ A_1+8A_2=0 \\ A_0-A_2=\frac{4}{\pi} \end{cases}$$

$$\implies A_1=A_2=0,\quad A_0=\frac{4}{\pi}$$

and the quadrature reads $\int_{-1}^{1} f(x)dx \approx \frac{4}{\pi} f(0)$.

2. a)

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(\xi_1)$$
$$f(x+2h) = f(x) + 2h f'(x) + \frac{(2h)^2}{2} f''(x) + \frac{(2h)^3}{6} f'''(\xi_2)$$

$$A f(x-h) + B f(x) + C f(x+2h) = f'(x) + \text{Error}.$$

$f(x)$: $A+B+C=0$

$f'(x)$: $h(-A+2C)=1$

$f''(x)$: $\frac{h^2}{2}(A+4C)=0$

$$\implies A=-4C \implies h\cdot 6C=1 \implies C=\frac{1}{6h}$$
$$\implies A=-\frac{4}{6h}=-\frac{2}{3h}$$

$$B=-(A+C)=-\left(-\frac{4}{6h}+\frac{1}{6h}\right)=\frac{3}{6h}=\frac{1}{2h}$$

And this is a formula of order $\mathcal{O}(h^2)$.

b) To approximate $f''(x)$:

$$A+B+C=0$$

$$h(-A+2C)=0 \Rightarrow A=2C$$

$$\frac{h^2}{2}(A+4C)=1 \Rightarrow 6C=\frac{2}{h^2} \Rightarrow C=\frac{1}{3h^2}$$

$$\Rightarrow A=\frac{2}{3h^2} \Rightarrow B=-(A+C)=-\frac{1}{h^2}.$$

This is a formula of order $O(h)$.

c) Add the answers from parts (a) and (b). The resulting formula is $O(h)$.

3.

$$\begin{pmatrix} 4 & 10 & 12 \\ 10 & 50 & 40 \\ 12 & 40 & 100 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 5 & 5 & 0 \\ 6 & 2 & \sqrt{60} \end{pmatrix} \begin{pmatrix} 2 & 5 & 6 \\ 0 & 5 & 2 \\ 0 & 0 & \sqrt{60} \end{pmatrix}$$

which can be obtained in different ways. The fastest is to assume directly the form

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}.$$

4. a. $T_2(x) = 2x^2 - 1 = 0 \Longrightarrow x_{0,1} = \pm\sqrt{\frac{1}{2}}$.

b. $P_1(x) = f(x_0)\dfrac{x-x_1}{x_0-x_1} + f(x_1)\dfrac{x-x_0}{x_1-x_0}$

$$= f\left(-\frac{1}{\sqrt{2}}\right)\frac{x-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}} + f\left(\frac{1}{\sqrt{2}}\right)\frac{x+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}.$$

Error $= \left|\dfrac{f''(\xi)}{2!}(x-x_0)(x-x_1)\right|$

An upper bound: $\dfrac{\max\left|f''(\xi)\right|}{4}$.

c. $\sin^{-1} 1 = \dfrac{\pi}{2}$

$\Rightarrow \pi = A + B$.

Also $0 = A\left(-\dfrac{1}{\sqrt{2}}\right) + B\left(\dfrac{1}{\sqrt{2}}\right)$

$\Longrightarrow A = B = \dfrac{\pi}{2}$.

Exact for polynomials in Π_3.

5. (a). $\varphi_0 = 1$

$\varphi_1 = x - c$

Orthogonality: $\langle \varphi_0, \varphi_1 \rangle_w = 0$.

$$\int_0^\infty e^{-x}(x-c)\,dx = \int_0^\infty x e^{-x}dx - c\int_0^\infty e^{-x}dx = 1 - c = 0$$

$\Rightarrow c = 1$ i.e. $\varphi_1 = x - 1$

(b) Let $p_1(x) = c_0\varphi_0 + c_1\varphi_1$ Let $f(x) = e^{-x}$.

Then $$c_0 = \frac{\langle f, \varphi_0\rangle_w}{\langle \varphi_0, \varphi_0\rangle_w} = \frac{\int_0^\infty e^{-x}\cdot 1 \cdot e^{-x}dx}{\int_0^\infty e^{-x}\,dx} = \frac{\frac{1}{2}}{1} = \frac{1}{2}.$$

$$c_1 = \frac{\langle f, \varphi_1\rangle}{\langle \varphi_1, \varphi_1\rangle_w} = \frac{\int_0^\infty e^{-x}(x-1)e^{-x}dx}{\int_0^\infty e^{-x}(x-1)^2dx} = \frac{-\frac{1}{4}}{1} = -\frac{1}{4}$$

$$\Rightarrow p_1(x) = \frac{1}{2}\varphi_0 - \frac{1}{4}\varphi_1 = \frac{1}{2}\cdot 1 - \frac{1}{4}(x-1) = \underline{-\frac{1}{4}x + \frac{3}{4}}$$