AMSC 466: Final Exam

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- The maximum grade is 100.
- Exam time: 2 hours
- Good luck!

Problems:

1. (a) (10 points). Using Taylor expansions for $f(x+h)$ and $f(x+k)$, derive the following approximation to $f'(x)$:

$$f'(x) \approx \frac{k^2 f(x+h) - h^2 f(x+k) + (h^2 - k^2) f(x)}{(k-h)kh}.$$

   (b) (5 points). Find the leading term in the approximation error.

2. (a) (10 points). Find a polynomial $p(x)$ of a minimal degree that satisfied

$$p(x_0) = f(x_0), \quad p(x_1) = 0, \quad p(x_2) = f(x_2), \quad p(x_3) = 0.$$

   (Assume that the points $\{x_i\}$ are distinct). Express the polynomial using Newton's form of the interpolating polynomial with divided differences. Evaluate the divided differences.

   (b) (5 points). Find an expression for the interpolation error. You may assume that the function $f(x)$ is continuously differentiable as many times as needed.

   (c) (5 points). Find a polynomial $p(x)$ of a minimal degree that satisfied

$$p(x_0) = f(x_0), \quad p'(x_0) = 0, \quad p(x_2) = f(x_2), \quad p'(x_2) = 0.$$

   (Assume that $x_0 \neq x_2$).

3. (a) (5 points). What is the purpose of the following iteration formula?

$$x_{n+1} = 2x_n - x_n^2 y.$$

   (b) (10 points). Write Newton's method for finding a root of the polynomial

$$p(x) = 4x^3 - 2x^2 + 3.$$

   Compute the first iteration x_1 if the starting point is $x_0 = -1$.

4. (a) (10 points). Find A, B, C such that the following approximation is exact for polynomials of degree ≤ 2. Explain why this formula is exact for *any* polynomial of degree ≤ 2.

$$\int_{-3h}^{h} f(x)dx \approx h[Af(0) + Bf(-h) + Cf(-2h)].$$

   (b) (10 points). Let $A_1(f, h)$ and $A_2(f, h)$ be two numerical integration methods that satisfy:

$$\int_a^b f(x)dx = A_1(f, h) + c_1 h + c_2 h^2 + \dots$$

$$\int_a^b f(x)dx = A_2(f, h) + 2c_1 h + c_3 h^3 + \dots$$

   Using this information, find a more accurate approximation of the intergal. What is the order of the approximation you found?

5. (a) (10 points). Let $w(x) = x^4$. Find the first three orthogonal polynomials with respect to the inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)w(x)dx.$$

   (Do not normalize the polynomials).

   (b) (10 points). Determine the nodes and weights for the Gaussian formula of the form

$$\int_{-1}^{1} x^4 f(x)dx \approx A_0 f(x_0) + A_1 f(x_1).$$

6. (10 points). Let A be the matrix:

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 9 \end{pmatrix}$$

   Explain why A should have a Cholesky decomposition and find it.