AMSC/CMSC 460
Final Exam – Solutions
Spring 2017

1) a) $P_1(x) = f(1)\ell_0(x) + f(1+h)\ell_1(x)$

$$\ell_0(x) = \frac{x-(1+h)}{-h}$$

$$\ell_1(x) = \frac{x-1}{h}.$$

b) $$\int_{1-h}^{1+h} f(x)dx \approx \int_{1-h}^{1+h} P_1(x)dx = f(1)\int_{1-h}^{1+h}\ell_0(x)dx + f(1+h)\int_{1-h}^{1+h}\ell_1(x)dx.$$

$$\int_{1-h}^{1+h}\ell_0(x)dx = \ldots = 2h.$$

$$\int_{1-h}^{1+h}\ell_1(x)dx = \ldots = 0.$$

$$\Rightarrow \int_{1-h}^{1+h} f(x)dx \approx 2hf(1).$$

2. a) $P_0 = 1$

$P_1 = x - c$

$$0 = \langle P_0, P_1 \rangle = \int_0^1 1\cdot(x-c) = \frac{x^2}{2} - cx \Big|_0^1 = \frac{1}{2} - c \Rightarrow c = \frac{1}{2}$$

To normalize:

$$\|P_0\|^2 = \int_0^1 P_0^2\, dx = 1$$

$$\|P_1\|^2 = \int_0^1 (x-\tfrac{1}{2})^2 dx = \int_0^1 (x^2 - x + \tfrac{1}{4}) dx = \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

$\Rightarrow$ The normalized polynomials are

$$\tilde{P}_0 = 1, \quad \tilde{P}_1(x) = \frac{x - 1/2}{\sqrt{1/12}}.$$

b) $Q_1(x) = c_0 P_0(x) + c_1 P_1(x)$.

$$c_0 = \frac{\langle f, P_0\rangle}{\|P_0\|^2} = \frac{\int_0^1 e^x dx}{1} = \frac{e-1}{1} = e - 1.$$

$$c_1 = \frac{\langle f, P_1\rangle}{\|P_1\|^2} = \frac{\int_0^1 e^x (x - 1/2) dx}{\int_0^1 (x-1/2)^2 dx} = \frac{1 - \frac{1}{2}(e-1)}{\frac{1}{12}} = 6(3-e)$$

$\Rightarrow Q_1(x) = e - 1 + 6(3-e)(x - 1/2)$.

c) $x_0 = \frac{1}{2}$ as the root of $P_1(x)$. $A_0 = 1$ since $1 = \int_0^1 1 dx = A_0$.
(We get for "free" that the quadrature is exact for x).

3) a) $\begin{cases} y'(t) = f(t, y(t)) \\ y(a) = y_0. \end{cases}$

The integral form of the ODE:

$$y(t+h) = y(t) + \int_t^{t+h} f(s, y(s))\,ds.$$

Replace the integral by a midpoint quadrature:

$$\int_t^{t+h} f(s, y(s))\,ds \approx h\, f\left(t+\frac{h}{2}, y\left(t+\frac{h}{2}\right)\right).$$

We obtain $y(t+\frac{h}{2})$ by predicting it using the Euler's method:

$$y\left(t+\frac{h}{2}\right) \approx y(t) + \frac{h}{2} f(t, y(t)).$$

$$\Rightarrow \begin{cases} w_{j+1/2} = w_j + \frac{h}{2} f(t_j, w_j) \\ w_{j+1} = w_j + h\, f(t_{j+1/2}, w_{j+1/2}). \end{cases}$$

b) $I = D(h) + c_1 h^2 + c_2 h^4 + O(h^6)$

$\Rightarrow I = D(2h) + c_1 (2h)^2 + c_2 (2h)^4 + O(h^6)$

$\Rightarrow I = \dfrac{4D(h) - D(2h)}{3} + O(h^4).$

with $D(h) = \dfrac{f(x-h) - 2f(x) + f(x+2h)}{h^2}$

and $D(2h) = \dfrac{f(x-2h) - 2f(x) + f(x+4h)}{4h^2}$

4. a) $P_3(x) = \underset{\overset{\shortparallel}{1}}{f[0]} + \underset{\overset{\shortparallel}{1}}{f[0,0]}\,x + f[0,0,1]x^2 + f[0,0,1,1]x^2(x-1)$

$$f[0,0,1] = \frac{f[0,1]-f[0,0]}{1-0} = \frac{\frac{f(1)-f(0)}{1} - 1}{1} = -1$$

$$f[0,1,1] = \frac{f'(1) - f[0,1]}{1} = 1$$

$$f[0,0,1,1] = \frac{f[0,1,1] - f[0,0,1]}{1-0} = \frac{1-(-1)}{1} = 2.$$

$$\Rightarrow P_3(x) = 1 + x - x^2 + 2x^2(x-1) = 1 + x - 3x^2 + 2x^3$$

b) $A = LL^T$ $\qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$