

## AMSC 460 - Midterm #2 - Solutions.

1. a) The table of divided differences corresponding to the data is:

$x$	$f(x)$	$f[x_i, j]$	$f[x_i, j, k]$	$f[x_i, j, k, l]$
$x_0 = 0$	1			
$x_1 = 1$	0	-1		
$x_2 = 2$	0	0	$\frac{1}{2}$	
$x_3 = 3$	0	0	0	$-\frac{1}{6}$

The Newton form of the interpolating polynomial is

$$\begin{aligned} P_3(x) &= f(x_0) + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) \\ &\quad + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2) \\ &= 1 - x + \frac{1}{2}x(x-1) - \frac{1}{6}x(x-1)(x-2). \end{aligned}$$

- b) Since there are 4 interpolation conditions, we are looking for a polynomial of degree  $\leq 3$ .

$$P_3(x) = f(0) + f[0, 0]x + f[0, 0, 0]x^2 + f[0, 0, 0, 1]x^2(x-1).$$

$$f(0) = 1$$

$$f[0, 0] = f'(0) = 0$$

$$f[0, 0, 0] = \frac{f[0, 1] - f[0, 0]}{1-0} = \frac{f(1) - f(0)}{1-0} - f'(0) = \frac{0-1}{1} - 0 = -1$$

$$f[0, 1, 1] = \frac{f[1, 1] - f[0, 1]}{1-0} = f'(1) - (-1) = 1.$$

$$f[0, 0, 1, 1] = \frac{f[0, 1, 1] - f[0, 0, 1]}{1-0} = \frac{1 - (-1)}{1} = 2$$

$$\Rightarrow \underline{P_3(x) = 1 - x^2 + 2x^2(x-1)}$$

2) a) Since in every subinterval the function  $f(x)$  is a cubic polynomial, all that we need to do is to verify that it is continuous as are its first and second derivatives.

Continuity of  $f(x)$  at  $x=0$ :  $-b=-d \Rightarrow \underline{b=d}$ .  
 " " " "  $x=1$   $\underline{c=e}$

$$f'(x) = \begin{cases} a + 3b(x-1)^2, & x \in (-\infty, 0], \\ c + 3d(x-1)^2, & x \in [0, 1], \\ e + 3f(x-1)^2, & x \in [1, \infty). \end{cases}$$

$$f'(0) : a + 3b = c + 3d \text{ but since } b=d, \text{ this implies } \underline{a=c}$$

$$f'(1) : \underline{c=e} \text{ (which we already know).}$$

$$f''(x) = \begin{cases} 6b(x-1) & x \in (-\infty, 0] \\ 6d(x-1) & x \in [0, 1] \\ 6f(x-1) & x \in [1, \infty). \end{cases}$$

$$f''(0) \Rightarrow -6b = -6d \Rightarrow b=d \text{ which we know.}$$

$$f''(1) \Rightarrow 0=0. \text{ Nothing new.}$$

Overall,  $f(x)$  is a cubic spline if  $\underline{a=c=e}$  and  $\underline{b=d}$ .

$$b) f(1) = c = 2 \Rightarrow a = c = e = 2.$$

$$-2 = f(0) = -b \Rightarrow b = 2 (=d)$$

$$f(2) = 7 = 2e + f = 4 + f \Rightarrow f = 3$$

$$\Rightarrow \underline{f(x) = \begin{cases} 2x + 2(x-1)^3, & x \in (-\infty, 0] \\ 2x + 2(x-1)^3, & x \in [0, 1] \\ 2x + 3(x-1)^3, & x \in [1, \infty) \end{cases}}$$

3) a) Set  $P_0 = 1$ .

Let  $P_1 = x - cP_0 = x - c$ .

$$\langle P_1, P_0 \rangle_w = \int_0^1 (x-c)e^x dx = e^x(x-1) - ce^x \Big|_0^1 = c(1-e) + 1 = 0$$

$$\Rightarrow c = \frac{1}{e-1} \Rightarrow \underline{P_1 = x - \frac{1}{e-1}}$$

b) Let  $\tilde{P}_0 = cP_0$

$$\text{Then } 1 = \|\tilde{P}_0\|_w^2 = \langle \tilde{P}_0, \tilde{P}_0 \rangle_w = \langle cP_0, cP_0 \rangle_w = c^2 \langle P_0, P_0 \rangle_w$$

$$= c^2 \int_0^1 e^x \cdot 1^2 dx = c^2 e^x \Big|_0^1 = c^2(e-1) \Rightarrow \underline{c = \frac{1}{\sqrt{e-1}}}$$

$$\text{(i.e. } \tilde{P}_0 = \frac{1}{\sqrt{e-1}} \text{)}$$

c) Let  $Q_1(x) = c_0 P_0(x) + c_1 P_1(x)$ .

Then

$$c_0 = \frac{\langle f, P_0 \rangle_w}{\langle P_0, P_0 \rangle_w} = \frac{\int_0^1 e^{-x} \cdot 1 \cdot e^x dx}{\int_0^1 1^2 e^x dx} = \frac{1}{e-1}$$

$$c_1 = \frac{\langle f, P_1 \rangle_w}{\langle P_1, P_1 \rangle_w} = \frac{\int_0^1 e^{-x} (x - \frac{1}{e-1}) e^x dx}{\int_0^1 (x - \frac{1}{e-1})^2 e^x dx} = \frac{\frac{1}{2} - \frac{1}{e-1}}{A}$$

$$A = \int_0^1 (x - \frac{1}{e-1})^2 e^x dx = \int_0^1 x^2 e^x dx - \frac{2}{e-1} \int_0^1 x e^x dx + \frac{1}{(e-1)^2} \int_0^1 e^x dx$$

$$= e^x(x^2 - 2x + 2) \Big|_0^1 - \frac{2}{e-1} (e^x(x+1)) \Big|_0^1 + \frac{1}{(e-1)^2} e^x \Big|_0^1$$

$$= e(1-2+2) - 2 - \frac{2}{e-1}(0 - (-1)) + \frac{1}{(e-1)^2}(e-1) = e - 2 - \frac{1}{e-1}$$