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AMSC 460 – FALL 2016 – SOLUTIONS

1) a) Use the method of undetermined coefficients.

$$f'(x) \approx A f(x-h) + B f(x) + C f(x+2h)$$

Write the Taylor expansions:

$$f(x-h) = f(x) - h f'(x) + \frac{1}{2} h^2 f''(x) - \frac{1}{6} h^3 f'''(x) + \dots$$

(We don't know how many terms are needed so we write these and if the 3^{rd} derivative ends cancelling out then we will add another term for the error).

$$f(x+2h) = f(x) + 2h f'(x) + \frac{1}{2}(2h)^2 f''(x) + \frac{1}{6}(2h)^3 f'''(x) + \dots$$

Now combine $A f(x-h) + B f(x) + C f(x+2h) =$

$$= A\left[f(x) - h f'(x) + \dots\right] + B f(x) + C\left[f(x) + 2h f'(x) + \dots\right]$$

$$= f(x)\left[A+B+C\right] + f'(x)\cdot h\left[-A+2C\right] + f''(x)\cdot \frac{h^2}{2}\left[A+4C\right]$$

$$+ f'''(x)\cdot \frac{h^3}{6}\left[-A+8C\right] + \dots$$

and we would like this to be $f'(x)$ (+error), which

implies that $\begin{cases} A+B+C=0 \\ (2C-A)h=1 \\ A+4C=0 \end{cases}$

$\Rightarrow A = -\frac{4}{6h}, \; B = \frac{3}{6h}, \; C = \frac{1}{6h}$

and the approximation is

$$f'(x) \approx \frac{1}{6h}\left[-4f(x-h) + 3f(x) + f(x+2h)\right].$$

To obtain the error term, we check the next term ($f'''(x)$).
Its coefficient is $(8C-A)\cdot\frac{h^3}{6}$ which is not zero for the A & C we calculated, hence it is the error term, and since A & C are both $O(\frac{1}{h})$, the error is $O(h^2)$.

b) We use the same expansions as in part (a) only this time we are approximating $f''(x) \approx Af(x-h) + Bf(x) + Cf(x+2h)$.

Then $\begin{cases} A+B+C=0 \\ 2C-A=0 \\ \frac{h^2}{2}(A+4C)=1 \end{cases} \Rightarrow A = \frac{2}{3h^2}, \; B = -\frac{1}{h^2}, \; C = \frac{1}{3h^2}.$

The approximation is then $f''(x) \approx \frac{1}{3h^2}\left[2f(x-h) - 3f(x) + f(x+2h)\right]$.
For the error, again $8C-A \neq 0$ which means that the next term is the error term. The h^3 is multiplied by $8C-A$ which in this case is $O(\frac{1}{h^2})$, hence the error is $O(h)$.

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2) a) $w(x)=x^2$, The interval is $[-1,1]$.

Then note that we are asked to find <u>two</u> orthogonal polynomials.

We set $P_0(x)=1$. So we have one of the two...

We need $P_1(x)$ which we set as $P_1(x)=x-c$.

We find c by requiring orthogonality:

$$0=\left(P_0,P_1\right)_w = \int_{-1}^{1} x^2\cdot 1\cdot(x-c)\,dx = \int_{-1}^{1}(x^3-cx^2)\,dx \Rightarrow c=0.$$

Hence $P_1(x)=x$.

b) $Q_0(x) = c_0P_0(x)$

$(f(x)=x^2)$

$$c_0 = \frac{\langle f(x), P_0(x)\rangle_w}{\|P_0(x)\|_w^2} = \frac{\int_{-1}^{1} \overset{f}{x^2}\cdot\overset{P_0}{1}\cdot\overset{w}{x^2}\,dx}{\int_{-1}^{1} \underset{P_0}{1}\cdot\underset{P_0}{1}\cdot\underset{w}{x^2}\,dx} = \frac{\left.\frac{x^5}{5}\right|_{-1}^{1}}{\left.\frac{x^3}{3}\right|_{-1}^{1}} = \frac{\frac{2}{5}}{\frac{2}{3}} = \frac{3}{5}.$$

$$\Rightarrow Q_0(x) = c_0P_0(x) = \frac{3}{5}\cdot 1 = \underline{\underline{\frac{3}{5}}}.$$

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3) a) We want a formula of the form $\int_{-1}^{1} f(x)dx \approx Af(0)+Bf(1)$.

We are asked to find such a quadrature through Lagrange interpolation. Hence we will interpolate $f(0)$ & $f(1)$:

$$P_1(x) = f(0)\frac{x-1}{0-1} + f(1)\frac{x-0}{1-0} = f(0)(1-x) + f(1)x.$$

The quadrature is obtained by integrating this linear polynomial:

$$\int_{-1}^{1} f(x)dx \approx \int_{-1}^{1} P_1(x)dx = f(0)\int_{-1}^{1}(1-x)dx + f(1)\int_{-1}^{1} x\,dx = \underline{\underline{2f(0)}}$$

[Note that this quadrature is exact for $f(x)=1$ and for $f(x)=x$ which means that it is exact for any polynomial of degree ≤ 1].

b) We are looking for a quadrature of the form

$$\int_0^1 f(x)dx \approx Af(0)+Bf(1)$$

that is exact $\forall$ functions of the form $f(x)=ax+b\sin(\pi x)$.

Hence we will try to find a quadrature that is <u>exact</u> for x and for $\sin(\pi x)$.

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$$\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} = A\cdot 0 + B\cdot 1 \implies B = \frac{1}{2}.$$

$$\int_0^1 \sin(\pi x)dx = -\frac{1}{\pi}\cos(\pi x)\Big|_0^1 = -\frac{1}{\pi}\left[\cos\pi - \cos 0\right] = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$$

$$= A\cdot \underbrace{\sin(\pi\cdot 0)}_{=0} + B\cdot \underbrace{\sin(\pi\cdot 1)}_{=0} = 0$$

So we get $\frac{2}{\pi} = 0$, which is nonsense.

Hence, no such quadrature exists.

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5) a) $f(x) = x^3+x-3$.

$f(x)$ is a continuous function.

$f(0) = -3 < 0$

$f(2) = 8+2-3 > 0$

$\Rightarrow$ By the IVT there exist at least one point $\xi \in (0,2)$ for which $f(\xi)=0$.

Newton's method: x_0

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \forall n \geq 0.$$

In our case $f(x) = x^3+x-3$
$f'(x) = 3x^2+1$

In which case $\begin{cases} x_0 \\ x_{n+1} = x_n - \dfrac{x_n^3+x_n-3}{3x_n^2+1} \end{cases} \forall n \geq 0.$

If $x_0 = 1$, then

$$x_1 = 1 - \frac{1+1-3}{3\cdot 1^2+1} = 1 - \frac{-1}{4} = \frac{5}{4}.$$

$$x_2 = 1 - \frac{\left(\frac{5}{4}\right)^3 + \frac{5}{4} - 3}{3\left(\frac{5}{4}\right)^2+1}.$$

b) $Q_3(x)$ is the given cubic polynomial due to the uniqueness of the interpolating polynomial.

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