CENTRE
DE RECHERCHES
MATHÉMATIQUES

AMSC/CMSC 460

MIDTERM #1 - SOLUTIONS

1) a) If we have an LU decomposition of A, i.e. $A=LU$ with L = lower triangular and U = upper triangular, solving $Ax=b$ is equivalent to solving $LUx=b$. We set $y=Ux$ and solve $Ly=b$ for y. Since L is lower triangular, $Ly=b$ can be easily solved using forward substitution. With the computed y, we can then solve $Ux=y$ for x, which can be easily done using backward substitution.

b) $$L=\begin{pmatrix} 1 & 0 & 0 \\ \frac{5}{4} & 1 & 0 \\ \frac{3}{2} & 2 & 1 \end{pmatrix} \qquad U=\begin{pmatrix} 8 & 10 & 12 \\ 0 & \frac{15}{2} & 15 \\ 0 & 0 & 52 \end{pmatrix}$$

c) Write the previous U as a product of a diagonal matrix and a unit upper triangular matrix:

$$U = \begin{pmatrix} 8 & 10 & 12 \\ 0 & \frac{15}{2} & 15 \\ 0 & 0 & 52 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & \frac{15}{2} & 0 \\ 0 & 0 & 52 \end{pmatrix} \begin{pmatrix} 1 & \frac{5}{4} & \frac{3}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Now break the diagonal matrix into the product of two identical diagonal matrices:

$$A = LU = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \frac{5}{4} & 1 & 0 \\ \frac{3}{2} & 2 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{8} & 0 & 0 \\ 0 & \sqrt{\frac{15}{2}} & 0 \\ 0 & 0 & \sqrt{52} \end{pmatrix}}_{\text{Define as } L} \underbrace{\begin{pmatrix} \sqrt{8} & 0 & 0 \\ 0 & \sqrt{\frac{15}{2}} & 0 \\ 0 & 0 & \sqrt{52} \end{pmatrix} \begin{pmatrix} 1 & \frac{5}{4} & \frac{3}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Define as } U = L^T}$$

i.e.

$$L = \begin{pmatrix} \sqrt{8} & 0 & 0 \\ \frac{5}{4}\sqrt{8} & \sqrt{\frac{15}{2}} & 0 \\ \frac{3}{2}\sqrt{8} & 2\sqrt{\frac{15}{2}} & \sqrt{52} \end{pmatrix}.$$

1. d. With scaled row pivoting, we have to decide in each step of the Gaussian elimination which row to use as the pivot row.

Starting with

$$\left(\begin{array}{ccc|c} 8 & 10 & 12 & 8 \\ 10 & 20 & 30 & 10 \\ 12 & 30 & 100 & 20 \end{array}\right)$$

we compute the ratio between the elements in the 1^{st} column and the maximum value in each row (all in absolute values). These values are $\frac{8}{12}, \frac{10}{30}, \frac{12}{100}$. Since $\frac{8}{12}$ is the largest, we keep the 1^{st} row as the pivot row and perform the 1^{st} stage of the elimination: $R_2 \to R_2 - \frac{10}{8}R_1$

$R_3 \to R_3 - \frac{12}{8}R_1$

$$\longrightarrow \left(\begin{array}{ccc|c} 8 & 10 & 12 & 8 \\ 0 & \frac{15}{2} & 15 & 0 \\ 0 & 15 & 82 & 8 \end{array}\right).$$

We check for the next pivot row. Since $\frac{\frac{15}{2}}{15} > \frac{15}{82}$ we keep the 2^{nd} row as the pivot row, and compute $R_3 \to R_3 - 2R_2$

$$\longrightarrow \left(\begin{array}{ccc|c} 8 & 10 & 12 & 8 \\ 0 & \frac{15}{2} & 15 & 0 \\ 0 & 0 & 52 & 8 \end{array}\right)$$

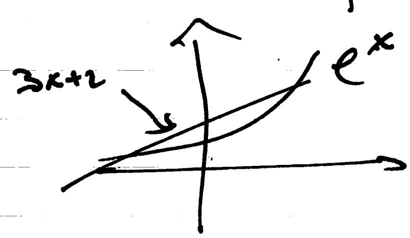
The solution can now be obtained with back substitution.

2. a. $f(x) = e^x - 3x - 2$.

$f(x)$ is continuous.

$f(0) = 1-2<0$, $f(10)>0$ $\Rightarrow$ by the midvalue THM there exists at least one root between 0 and 10.

b. A root of $f(x)$ satisfies $e^x = 3x+2$



These functions intersect twice but only once for positive x's $\Rightarrow$ Exactly <u>one</u> positive root.

c. Newton's method: Choose x_0.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} - 3x_n - 2}{e^{x_n} - 3}.$$

d. The secant method: Choose x_0, x_1

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}} = x_n - \frac{(x_n - x_{n-1})(e^{x_n} - 3x_n - 2)}{(e^{x_n} - 3x_n - 2) - (e^{x_{n-1}} - 3x_{n-1} - 2)}$$

The secant method does not require to compute the derivative of $f(x)$. It has a slower convergence rate than Newton's method.

3\. a. 25 in binary is $11001 = 2^0 + 2^3 + 2^4 = 1+8+16$

0.22 in binary:

$.22 \times 2 = .44$
$.44 \times 2 = .88$
$.88 \times 2 = 1.76$
$.76 \times 2 = 1.52$
$.52 \times 2 = 1.04$
$.04 \times 2 = .08$
$.08 \times 2 = .16$
$.16 \times 2 = .32$
$.32 \times 2 = .64$
$.64 \times 2 = 1.28$
and so on...

$\Rightarrow (25.22)_2 = 11001.0011100001\ldots$

b. Approach I: Use the leading bit for the sign: $\overset{\text{"}-\text{"}}{1}0011001$

Approach II: 2's complement: $(-25)_2 = 11100111$

c. $(25.22)_2 = 1.\underbrace{10010011100001\ldots}\times 2^{\overset{E}{4}}$

$\Rightarrow \underset{\text{Sign}}{0} \,|\, \underbrace{00000100}_{E=4} \,|\, 10010011100001\ldots$