AMSC/CMSC 460: Final Exam - SOLUTIONS

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 2 hours.
- Good luck!

Additional instructions:

- The exam has 2 parts: part A and part B. Each part has 4 problems.
- You should solve only 3 out of the 4 problems in each part.
- No extra credit will be given for solving more than 3 problems in each part.
- If you solve more than 3 problems, you should clearly indicate which problems you would like to be graded - otherwise, the first 3 problems in each part will be graded.

Part A: Choose 3 problems out of problems 1-4 (Each problem = 10 points)

1. Find the most accurate approximation to $f'(x)$ using $f(x-\frac{h}{2}), f(x), f(x+h)$. What is the order of accuracy of this approximation?

   Solution: We would like to approximate

   $$f'(x) \approx Af\left(x-\frac{h}{2}\right)+Bf(x)+Cf(x+h).$$

   We write the Taylor expansions for each of the terms:

   $$\begin{aligned} f\left(x-\frac{h}{2}\right) &= f(x)-\frac{h}{2}f'(x)+\frac{1}{2}\left(\frac{h}{2}\right)^2 f''(x)-\frac{1}{6}\left(\frac{h}{2}\right)^3 f'''(x)+\dots \\ f(x+h) &= f(x)+hf'(x)+\frac{1}{2}h^2 f''(x)+\frac{1}{6}h^3 f'''(x)+\dots \end{aligned}$$

   Hence, by setting the coefficients of $f(x)$ and $f''(x)$ in the expansion to zero, and the coefficient of $f'(x)$ to 1, we get the following linear system:

   $$\begin{cases} A+B+C=0, \\ -\dfrac{h}{2}A+hC=1, \\ \dfrac{h^2}{2}\left(\dfrac{A}{4}+C\right)=0. \end{cases}$$

   The solution of this system is:

   $$A=-\frac{4}{3h}, \quad B=\frac{1}{h}, \quad C=\frac{1}{3h}.$$

   Hence, the approximation is

   $$f'(x) \approx \frac{-4f\left(x-\frac{h}{2}\right)+3f(x)+f(x+h)}{3h}.$$

   The order of the approximation is $O(h^2)$ since the next term in the Taylor expansion does not vanish, and the h^3 in front of the $f'''(x)$ term is to be divided by h since this term is multiplied by A and by C, both which are $O(1/h)$.

2. Find a quadrature of the form

   $$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = A_0 f(x_0)+A_1 f(x_1),$$

   that is exact for all polynomials of degree ≤ 3.

Solution: This is a Gaussian quadrature. The given interval $[-1,1]$ and weight, $1/\sqrt{1-x^2}$, correspond to Chebyshev polynomials. Since the quadrature is based on two points, x_0 and x_1 are the roots of the quadratic Chebyshev polynomial, $T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1$. This means that the quadrature points are

$$x_{0,1} = \pm\frac{1}{\sqrt{2}}.$$

Once the quadrature points are known, all that remains is to find the coefficients, A_0 and A_1. We do this through the method of undetermined coefficients. FIrst, we require that the quadrature is exact for $f(x) = 1$, i.e.,

$$\pi = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = A_0 + A_1.$$

We also require that the quadrature is exact for $f(x) = x$, i.e.,

$$0 = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}dx = -\frac{A_0}{\sqrt{2}} + \frac{A_1}{\sqrt{2}}.$$

Hence $A_0 = A_1 = \pi/2$, and the desired quadrature is

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx \approx \frac{\pi}{2}\left(f\left(-\frac{1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{2}}\right)\right).$$

3. (a) Write the Lagrange form of the linear interpolation polynomial that interpolates $f(x)$ at $x = -1, 1$.

Solution:

$$\begin{aligned} P_1(x) &= f(-1)\frac{x-1}{-1-1} + f(1)\frac{x+1}{1+1} = \\ &= f(-1)\frac{x-1}{-2} + f(1)\frac{x+1}{2} = \\ &= x\frac{f(1)-f(-1)}{2} + \frac{f(1)+f(-1)}{2}. \end{aligned}$$

(b) Use the interpolant you obtained in part (a) to find a weighted quadrature of the form

$$\int_{-2}^{2} xf(x)dx = A_0 f(-1) + A_1 f(1).$$

Solution:

We approximate

$$\begin{aligned}\int_{-2}^{2} xf(x)dx &\approx \int_{-2}^{2} xP_1(x)dx = \\ &= \int_{-2}^{2}\left(x^2\frac{f(1)-f(-1)}{2}+x\frac{f(1)+f(-1)}{2}\right)dx = \dots \\ &= \frac{8}{3}(f(1)-f(-1)).\end{aligned}$$

Hence, the quadrature coefficients are $A_0 = \frac{8}{3}$ and $A_1 = -\frac{8}{3}$.

4. Find a linear polynomial, $P_1^*(x)$, that minimizes

$$\int_{-\infty}^{\infty} e^{-x^2}(x^3 - Q_1(x))^2 dx,$$

among all polynomials $Q_1(x)$ of degree ≤ 1.

Solution:

This is a least squares problem with Hermite polynmials (the weight is $w(x) = e^{-x^2}$ and the interval is $(-\infty, \infty)$. With $H_0(x) = 1$ and $H_1(x) = 2x$, the solution is given by

$$P_1^*(x) = c_0 H_0(x) + c_1 H_1(x),$$

with

$$c_0 = \frac{\langle x^3, H_0\rangle_w}{\|H_0\|_w^2} = \frac{\int_{-\infty}^{\infty} e^{-x^2} x^3 dx}{\dots} = 0,$$

and

$$c_1 = \frac{\langle x^3, H_1\rangle_w}{\|H_1\|_w^2} = \frac{\int_{-\infty}^{\infty} e^{-x^2} x^4 dx}{\int_{-\infty}^{\infty} (H_1(x))^2 e^{-x^2} dx} = \frac{2\Gamma(5/2)}{2\sqrt{\pi}} = \frac{3}{4}.$$

Hence the solution is

$$P_1^*(x) = c_1 H_1(x) = \frac{3}{2}x.$$

Part B: Choose 3 problems out of problems 5-8 (Each problem = 10 points)

5. Find values for a, b, c, d such that the following function, $s(x)$, is a cubic spline on $[0, 2]$ that satisfies $s'(2) = 0$,

$$s(x) = \begin{cases} x^3 - ax^2 + b, & 0 \le x \le 1, \\ cx^3 + dx^2, & 1 \le x \le 2. \end{cases}$$

Solution: We start by computing the first and second derivatives of $s(x)$:

$$s'(x) = \begin{cases} 3x^2 - 2ax, & 0 \le x \le 1, \\ 3cx^2 + 2dx, & 1 \le x \le 2. \end{cases}$$

$$s''(x) = \begin{cases} 6x - 2a, & 0 \le x \le 1, \\ 6cx + 2d, & 1 \le x \le 2. \end{cases}$$

The continuity of $s(x)$ at $x = 1$ implies

$$1 - a + b = c + d.$$

The continuity of $s'(x)$ at $x = 1$ implies

$$3 - 2a = 3c + 2d.$$

The continuity of $s''(x)$ at $x = 1$ implies

$$6 - 2a = 6c + 2d.$$

Requiring that $s'(2) = 0$ implies that

$$12c + 4d = 0.$$

The solution of the linear system is: $a = 3$, $b = 0$, $c = 1$, and $d = -3$, which means that the spline is simply a cubic polynomial

$$s(x) = x^3 - 3x^2, \quad 0 \le x \le 2.$$

6. Use the Gram-Schmidt process to find <u>orthonormal</u> polynomials of degrees 0 and 1 with respect to the inner product

$$\langle f, g \rangle_w = \int_0^\infty f(x)g(x)e^{-2x}dx.$$

Solution: We will first compute the polynomials without normalizing them. At the end we will compute the normalization constants. We start with the constant polynomial, $P_0(x) = 1$. We then set

$$P_1(x) = x - cP_0 = x - c,$$

and compute c such that P_1 is orthogonal to P_0:

$$\begin{aligned} 0 &= \langle P_0, P_1 \rangle_w = \int_0^\infty 1(x-c)e^{-2x}dx = \int_0^\infty xe^{-2x}dx - c\int_0^\infty e^{-2x}dx = \\ &= \frac{e^{-2x}}{-2}\left(x - \frac{1}{-2}\right)\Bigg|_0^\infty - c\frac{e^{-2x}}{-2}\Bigg|_0^\infty = \frac{1}{2}(\frac{1}{2} - c). \end{aligned}$$

Hence $c = 1/2$, and therefore $P_1(x) = x - \frac{1}{2}$. We now normalize $P_0(x)$ and $P_1(x)$. We set $\tilde{P}_0 = cP_0$. Then

$$1 = \left\langle \tilde{P}_0, \tilde{P}_0 \right\rangle_w = c^2 \langle 1, 1 \rangle_w = c^2 \int_0^\infty e^{-2x}dx = c^2 \frac{e^{-2x}}{-2}\Bigg|_0^\infty = \frac{c^2}{2}.$$

Hence $c = \sqrt{2}$ and $\tilde{P}_0 = \sqrt{2}$. Setting $\tilde{P}_1(x) = cP_1(x)$ we have

$$1 = \left\langle \tilde{P}_1, \tilde{P}_1 \right\rangle_w = c^2 \left\langle x - \frac{1}{2}, x - \frac{1}{2} \right\rangle_w = c^2 \int_0^\infty \left(x - \frac{1}{2}\right)^2 e^{-2x}dx = \ldots = \frac{c^2}{8}.$$

Hence $c = \sqrt{8}$ and $\tilde{P}_1(x) = \sqrt{8}(x - 1/2)$.

7. Explain what the floating point representation of $\frac{1}{10}$ looks like on a 32-bit machine.

Solution:

Write 1/10 in base 2:

$$\left(\frac{1}{10}\right)_2 = 0.0001100110011\ldots = 1.100110011\ldots \times 2^{-4}.$$

We divide the 32 bits into 3 parts: 1 bit for the sign. In this case since the number is positive we will use 0. The next 8 bits are for the exponent. In this case the exponent is -4, which we will write using 2's complement as 11111100 (this should be explained). Finally, we use the remaining 23 bits to represent the mantissa. We skip the leading "1" and write 10011001100110011001100. Overall, the number -4 is stored as:

$$0\Big|11111100\Big|10011001100110011001100.$$

8. Find a Cholesky decomposition of

$$A = \begin{pmatrix} 16 & 12 & 4 \\ 12 & 13 & 3 \\ 4 & 3 & 17 \end{pmatrix}.$$

Solution: Using standard techniques we find a lower triangular matrix

$$L = \begin{pmatrix} 4 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix},$$

such that $LL^T = A$.

- Chebyshev polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) = 0, \ \forall n \geq 1.$$

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = 0, \quad m \neq n.$$

$$\int_{-1}^{1} \frac{(T_n(x))^2}{\sqrt{1-x^2}} dx = \begin{cases} \pi, & n = 0, \\ \frac{\pi}{2}, & n = 1, 2, \dots \end{cases}$$

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi.$$

- Hermite polynomials

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \ \forall n \geq 1$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x) = \delta_{nm} 2^n n! \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = \Gamma\left(\frac{m+1}{2}\right), \quad \text{for even } m$$

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(3/2) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma(5/2) = \tfrac{3}{4}\sqrt{\pi}.$$

- Other formulas

$$\int xe^{ax} dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right).$$

$$\int x^2 e^{ax} dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right).$$