AMSC/CMSC 460: Midterm 2

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April 17, 2018

Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 75 minutes
- Good luck!

Problems: (Each problem = 10 points)

1. (a) Explain the advantages of interpolating at Chebyshev points.

Solution: When interpolating data that is sampled from a function $f(x)$ at $n+1$ points $x_0, \dots, x_n$, the interpolation error is given by

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i).$$

When considering the interval $[-1,1]$, choosing $x_0, \dots, x_n$ as Chebyshev points (i.e. the $n+1$ roots of $T_{n+1}(x)$, minimizes the product term in the error, i.e.,

$$\max_{x\in[-1,1]}\prod_{i=0}^{n}(x-x_i),$$

has the smallest value out of all possible choices of interpolation points.

(b) Compute the unique interpolating polynomial of degree ≤ 2 that interpolates data sampled from $f(x) = x^2$ at an appropriate number of Chebyshev points on the interval $[-1,1]$.

Solution: Since we are asked to compute a quadratic polynomial that interpolates data that is sampled from a quadratic function, we can immediately use the uniqueness of the interpolating polynomial and conclude that the answer must be the function itself, i.e., $P_2(x) = x^2$. Any direct calculation must lead to this answer.

(c) Repeat part (b) with $f(x) = x^4$.

Solution: Here we should compute the answer. We are seeking a quadratic interpolant, which means that we need 3 values. The 3 values should be the roots of the cubic Chebyshev polynomial $T_3(x) = 4x^3 - 3x$:

$$x_0 = -\frac{\sqrt{3}}{2}, \quad x_1 = 0, \quad x_2 = \frac{\sqrt{3}}{2}.$$

Computing the divided differences, we get

$$f(x_0) = \frac{9}{16}, \quad f[x_0, x_1] = -\frac{9}{8\sqrt{3}}, \quad f[x_0, x_1, x_2] = \frac{3}{4}.$$

Hence, the interpolating polynomial is

$$P_2(x) = \frac{9}{16} - \frac{9}{8\sqrt{3}}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4}\left(x + \frac{\sqrt{3}}{2}\right)x = \dots = \frac{3}{4}x^2.$$

Note: Chebyshev polynomials are given by

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) = 0, \forall n \geq 1.$$

2. Find a spline of degree 2, $S(x)$, on the interval $[0, 2]$, for which $S(0) = 0$, $S(1) = 2$, $S(2) = 0$, and $S'(0) = 0$. Use the points $0, 1, 2$ as the knots.

Solution:

Consider a quadratic spline of the form

$$S(x) = \begin{cases} S_0(x), & 0 \leq x \leq 1, \\ S_1(x), & 1 \leq x \leq 2. \end{cases} = \begin{cases} a_0 + a_1 x + a_2 x^2, & 0 \leq x \leq 1, \\ b_0 + b_1 x + b_2 x^2, & 1 \leq x \leq 2. \end{cases}$$

We write 6 equations for the 6 unknowns. First we have the 4 interpolation conditions:

- $S_0(0) = 0 \Longrightarrow a_0 = 0.$
- $S_0(1) = 2 \Longrightarrow a_0 + a_1 + a_2 = 2.$
- $S_1(1) = 2 \Longrightarrow b_0 + b_1 + b_2 = 2.$
- $S_1(2) = 0 \Longrightarrow b_0 + 2b_1 + 4b_2 = 0.$

The continuity of the first derivative implies that

$$S_0'(1) = S_1'(1) \Longrightarrow a_1 + 2a_2 = b_1 + 2b_2.$$

Finally, we have the additional condition at the first derivative:

$$S'(0) = 0 \Longrightarrow a_1 = 0.$$

Solving for $a_0, a_1, a_2, b_0, b_1, b_2$ we end up with

$$S(x) = \begin{cases} 2x^2, & 0 \leq x \leq 1, \\ -8 + 16x - 6x^2, & 1 \leq x \leq 2. \end{cases}$$

3. Use the Gram-Schmidt process to find orthogonal polynomials of degrees 0, 1, 2, on the interval $[0, 1]$, with respect to the weight $w(x) = 1 + x$.

Note: you do not need to normalize the polynomials. For the quadratic polynomial, $P_2(x)$, write the coefficients but do not explicitly calculate the integrals.

Solution: We note that we are only asked to find orthogonal polynomials without normalizing them. Set $P_0(x) = 1$. We then let $P_1(x) = x - cP_0(x)$. The orthogonality condition, $\langle P_0, P_1 \rangle_w = 0$, implies that $\langle P_0, x - cP_0 \rangle_w =$, i.e., $\langle P_0, x \rangle_w = c \langle P_0, P_0 \rangle_w$, or

$$c = \frac{\langle P_0, x \rangle_w}{\|P_0\|_w^2} = \frac{\int_0^1 x \cdot 1 \cdot (1 + x) dx}{\int_0^1 1^2 (1 + x) dx} = \ldots = \frac{5}{9}.$$

Hence $P_1(x) = x - \frac{5}{9}$. For the quadratic polynomial, we set

$$P_2(x) = x^2 - cP_0(x) - dP_1(x).$$

The orthogonality condition $\langle P_0, P_2 \rangle_w = 0$ implies that

$$c = \frac{\langle x^2, P_0 \rangle_w}{\|P_0\|_w^2} = \frac{\int_0^1 x^2 \cdot 1 \cdot (1+x)dx}{\frac{3}{2}}$$

The orthogonality condition $\langle P_1, P_2 \rangle_w = 0$ implies that

$$d = \frac{\langle x^2, P_1 \rangle_w}{\|P_1\|_w^2} = \frac{\int_0^1 x^2(x - \frac{5}{9})(x+1)dx}{\int_0^1 (x - \frac{5}{9})^2(x+1)dx}.$$

4. Let $f(x) = x^2$. Find the quadratic polynomial $Q_2^*(x)$ that minimizes

$$\int_{-\infty}^{\infty} e^{-x^2}(f(x) - Q_2(x))^2 dx,$$

among all quadratic polynomials $Q_2(x)$.

Note: You may use:

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \forall n \geq 1$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x) = \delta_{nm}2^n n! \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = \Gamma\left(\frac{m+1}{2}\right), \quad \text{for even } m$$

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(3/2) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma(5/2) = \tfrac{3}{4}\sqrt{\pi}.$$

Solution: We have the appropriate orthogonal polynomials:

$$H_0(1) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2.$$

We note that the norms are

$$\|H_0\|^2 = \sqrt{\pi}, \quad \|H_1\|^2 = 2\sqrt{\pi}, \quad \|H_2\|^2 = 8\sqrt{\pi}.$$

Set

$$Q_2^*(x) = c_0H_0(x) + c_1H_1(x) + c_2H_2(x).$$

Then

$$c_0 = \frac{\langle f, H_0 \rangle_w}{\|H_0\|_w^2} = \frac{\int_{-\infty}^{\infty} e^{-x^2} x^2 dx}{\sqrt{\pi}} = \frac{\frac{\sqrt{\pi}}{2}}{\sqrt{\pi}} = \frac{1}{2}.$$

$$c_1 = \frac{\langle f, H_1 \rangle_w}{\|H_1\|_w^2} = \frac{\int_{-\infty}^{\infty} e^{-x^2} x^2 \cdot 2x dx}{2\sqrt{\pi}} = 0.$$

$$c_2 = \frac{\langle f, H_2 \rangle_w}{\|H_2\|_w^2} = \frac{\int_{-\infty}^{\infty} e^{-x^2} x^2 (4x^2 - 2) dx}{8\sqrt{\pi}} = \frac{4\Gamma(\frac{5}{2}) - 2\Gamma(\frac{3}{2})}{8\sqrt{\pi}} = \frac{4\frac{3}{4}\sqrt{\pi} - 2\frac{1}{2}\sqrt{\pi}}{8\sqrt{\pi}} = \frac{1}{4}.$$

Hence

$$Q_2^*(x) = \frac{1}{2}H_0(x) + \frac{1}{4}H_2(x) = \frac{1}{2} + \frac{1}{4}(4x^2 - 2) = x^2$$

as should be...