

## AMSC 460 - FALL 2016 - SOLUTIONS

1) a) Use the method of undetermined coefficients.

$$f'(x) \approx Af(x-h) + Bf(x) + Cf(x+2h)$$

Write the Taylor expansions:

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2f''(x) - \frac{1}{6}h^3f'''(x) + \dots$$

(We don't know how many terms are needed so we write these and if the 3<sup>rd</sup> derivative ends canceling out then we will add another term for the error).

$$f(x+2h) = f(x) + 2hf'(x) + \frac{1}{2}(2h)^2f''(x) + \frac{1}{6}(2h)^3f'''(x) + \dots$$

Now combine  $Af(x-h) + Bf(x) + Cf(x+2h) =$

$$= A[f(x) - hf'(x) + \dots] + Bf(x) + C[f(x) + 2hf'(x) + \dots]$$

$$= f(x)[A+B+C] + f'(x) \cdot h[-A+2C] + f''(x) \cdot \frac{h^2}{2}[A+4C]$$

$$+ f'''(x) \cdot \frac{h^3}{6}[-A+8C] + \dots$$

and we would like this to be  $f'(x)$  (+error), which

implies that

$$\begin{cases} A+B+C=0 \\ (2C-A)h=1 \\ A+4C=0 \end{cases}$$

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$$\Rightarrow A = -\frac{4}{6h}, B = \frac{3}{6h}, C = \frac{1}{6h}$$

and the approximation is

$$f'(x) \approx \frac{1}{6h} [-4f(x-h) + 3f(x) + f(x+2h)].$$

To obtain the error term, we check the next term ( $f'''(x)$ ).

Its coefficient is  $(8C-A) \cdot \frac{h^3}{6}$  which is not zero for the  $A \neq C$

we calculated, hence it is the error term, and since  $A \neq C$

are both  $O(\frac{1}{h})$ , the error is  $O(h^2)$ .

b) We use the same expansions as in part (a) only this

time we are approximating  $f''(x) \approx Af(x-h) + Bf(x) + Cf(x+2h)$ .

$$\text{Then } \begin{cases} A+B+C=0 \\ 2C-A=0 \\ \frac{h^2}{2}(A+4C)=1 \end{cases} \Rightarrow A = \frac{2}{3h^2}, B = -\frac{1}{h^2}, C = \frac{1}{3h^2}.$$

The approximation is then  $f''(x) \approx \frac{1}{3h^2} [2f(x-h) - f(x) + f(x+2h)]$ .

For the error, again  $8C-A \neq 0$  which means that the next

term is the error term. The  $h^3$  is multiplied by  $8C-A$  which

in this case is  $O(\frac{1}{h^2})$ , hence the error is  $O(h)$ .

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2) a)  $w(x) = x^2$ , The interval is  $[-1, 1]$ .

Then note that we are asked to find two orthogonal polynomials.

We set  $P_0(x) = 1$ . So we have one of the two...

We need  $P_1(x)$  which we set as  $P_1(x) = x - c$ .

We find  $c$  by requiring orthogonality:

$$0 = \langle P_0, P_1 \rangle_w = \int_{-1}^1 x^2 \cdot 1 \cdot (x - c) dx = \int_{-1}^1 (x^3 - cx^2) dx \Rightarrow c = 0.$$

Hence  $P_1(x) = x$ .

b)  $Q_0(x) = c_0 P_0(x)$

$$c_0 = \frac{\langle f(x), P_0(x) \rangle_w}{\|P_0(x)\|_w^2} = \frac{\int_{-1}^1 x^2 \cdot 1 \cdot x^2 dx}{\int_{-1}^1 1 \cdot 1 \cdot x^2 dx} = \frac{\frac{x^5}{5} \Big|_{-1}^1}{\frac{x^3}{3} \Big|_{-1}^1} = \frac{\frac{1}{5} - \frac{-1}{5}}{\frac{1}{3} - \frac{-1}{3}} = \frac{\frac{2}{5}}{\frac{2}{3}} = \frac{3}{5}.$$

$\begin{matrix} (f(x)=x^2) & & f & P_0 & w \\ & \searrow & \searrow & \searrow & \\ & & x^2 & 1 & x^2 \\ & & \uparrow & \uparrow & \uparrow \\ & & P_0 & P_0 & w \end{matrix}$

$$\Rightarrow Q_0(x) = c_0 P_0(x) = \frac{3}{5} \cdot 1 = \frac{3}{5}.$$

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3) a) We want a formula of the form  $\int_{-1}^1 f(x) dx \approx Af(0) + Bf(1)$ .

We are asked to find such a quadrature through Lagrange interpolants. Hence we will interpolate  $f(0)$  &  $f(1)$ :

$$P_1(x) = f(0) \frac{x-1}{0-1} + f(1) \frac{x-0}{1-0} = f(0)(1-x) + f(1)x.$$

The quadrature is obtained by integrating this linear polynomial:

$$\int_{-1}^1 f(x) dx \approx \int_{-1}^1 P_1(x) dx = f(0) \int_{-1}^1 (1-x) dx + f(1) \int_{-1}^1 x dx = \underline{\underline{2f(0)}}$$

[Note that this quadrature is exact for  $f(x)=1$  and for  $f(x)=x$  which means that it is exact for any polynomial of degree  $\leq 1$ .]

b) We are looking for a quadrature of the form

$$\int_0^1 f(x) dx \approx Af(0) + Bf(1)$$

that is exact for functions of the form  $f(x) = ax + b \sin(\pi x)$ .

Hence we will try to find a quadrature that is exact for  $x$  and for  $\sin(\pi x)$ .

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$$\int_0^1 x \, dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2} = A \cdot 0 + B \cdot 1 \Rightarrow B = \frac{1}{2}.$$

$$\int_0^1 \sin(\pi x) \, dx = -\frac{1}{\pi} \cos(\pi x) \Big|_0^1 = -\frac{1}{\pi} [\cos \pi - \cos 0] = -\frac{1}{\pi} (-1 - 1) = \frac{2}{\pi}$$

$$= A \cdot \underbrace{\sin(\pi \cdot 0)}_0 + B \cdot \underbrace{\sin(\pi \cdot 1)}_0 = 0$$

So we get  $\frac{2}{\pi} = 0$ , which is nonsense.

Hence, no such quadrature exists.

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$$5) a) f(x) = x^3 + x - 3.$$

$f(x)$  is a continuous function.

$$f(0) = -3 < 0$$

$$f(2) = 8 + 2 - 3 > 0$$

$\Rightarrow$  By the IVT there exist at least one point  $\xi \in (0, 2)$   
for which  $f(\xi) = 0$ .

Newton's method:  $x_0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \forall n \geq 0.$$

In our case  $f(x) = x^3 + x - 3$   
 $f'(x) = 3x^2 + 1$

In which case  $\begin{cases} x_0 \\ x_{n+1} = x_n - \frac{x_n^3 + x_n - 3}{3x_n^2 + 1} \end{cases} \quad \forall n \geq 0.$

If  $x_0 = 1$ , then

$$x_1 = 1 - \frac{1 + 1 - 3}{3 \cdot 1^2 + 1} = 1 - \frac{-1}{4} = \frac{5}{4}.$$

$$x_2 = 1 - \frac{\left(\frac{5}{4}\right)^3 + \frac{5}{4} - 3}{3\left(\frac{5}{4}\right)^2 + 1}.$$

b)  $Q_3(x)$  is the given cubic polynomial due to the uniqueness of the interpolating polynomial.