AMSC/CMSC 460: Midterm 2 Solutions

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 75 minutes
- Good luck!

Problems:

1. **(10 points)** Using Newton's form of the Hermite interpolation polynomial, find the polynomial of degree ≤ 3 that interpolates: $f(1) = 0$, $f'(1) = 2$, $f(2) = 1$, $f'(2) = 3$. (Compute explicitly all the divided differences).

   Solution:

   The polynomial we are seeking is of the form:

   $$Q_2(x) = f[1] + f[1,1](x-1) + f[1,1,2](x-1)^2 + f[1,1,2,2](x-1)^2(x-2).$$

   The divided differences are: $f[1] = f(1) = 0$. $f[1,1] = f'(1) = 2$. Also

   $$f[1,1,2] = \frac{f[1,2] - f[1,1]}{2-1} = \frac{\frac{f(2)-f(1)}{2-1} - f'(1)}{1} = \frac{\frac{1-0}{1} - 2}{1} = -1.$$

   $$f[1,2,2] = \frac{f[2,2] - f[1,2]}{2-1} = f'(2) - \frac{f(2) - f(1)}{2-1} = 3 - 1 = 2$$

   $$f[1,1,2,2] = \frac{f[1,2,2] - f[1,1,2]}{2-1} = 2 - (-1) = 3.$$

   Hence

   $$Q_2(x) = 2(x-1) - (x-1)^2 + 3(x-1)^2(x-2).$$

2. Let $w(x) = 1$, $\forall x \in [-3, 2]$.

   (a) **(10 points)** Find the first two orthogonal polynomials with respect to the inner product

   $$\langle f(x), g(x) \rangle_w = \int_{-3}^{2} f(x)g(x)w(x)dx.$$

   Solution: Let $P_0(x) = 1$. Set $P_1(x) = x - cP_0(x) = x - c$. Then

   $$\langle P_0, P_1 \rangle = \int_{-3}^{2} (x-c)dx = \frac{x^2}{2} - cx \Big|_{-3}^{2} = -\frac{5}{2} - 5c = 0,$$

   Hence $c = -\frac{1}{2}$, which means that $P_1(x) = x + \frac{1}{2}$.

   (b) **(10 points)** Normalize the polynomials you found in part (a).

Solution:

$$\|P_0(x)\|^2 = \langle P_0, P_0 \rangle = \int_{-3}^{2} dx = 5.$$

Hence, if we set $\tilde{P}_0 = cP_0$, then

$$\|\tilde{P}_0\| = 1 = c^2\|P_0\|^2 = 5c^2.$$

i.e., $c = \frac{1}{\sqrt{5}}$, which means that $\tilde{P}_0(x) = \frac{1}{\sqrt{5}}$.
Similarly,

$$\begin{aligned}\|P_1(x)\|^2 &= \langle P_1, P_1 \rangle = \int_{-3}^{2} \left(x + \frac{1}{2}\right)^2 dx = \int_{-3}^{2} \left(x + x + \frac{1}{4}\right) dx = \\ &= \frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{4}x\Big|_{-3}^{2} = \frac{8}{3} + \frac{4}{2} + \frac{2}{4} - \left(-\frac{27}{3} + \frac{9}{2} - \frac{3}{4}\right) = \frac{125}{12}\end{aligned}$$

Hence, the normalized $\tilde{P}_1(x)$ is

$$\tilde{P}_1(x) = \sqrt{\frac{12}{125}}\left(x + \frac{1}{2}\right).$$

(c) **(10 points)** Find the polynomial of degree 0, $Q_0(x)$, that minimizes

$$\int_{-3}^{2} (e^x - Q_0(x))^2 dx.$$

Solution: We let $Q_0(x) = c_0\tilde{P}_0(x)$, where $\tilde{P}_0 = \frac{1}{\sqrt{5}}$ is the orthonormal polynomial of degree 0 computed in part (b). Since this is a normalized polynomial, we can write

$$c_0 = \left\langle f, \tilde{P}_0 \right\rangle = \int_{-3}^{2} \frac{e^x}{\sqrt{5}} dx = \frac{e^2 - e^{-3}}{\sqrt{5}}.$$

This means that

$$Q_0(x) = \frac{e^2 - e^{-3}}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} = \frac{e^2 - e^{-3}}{5}.$$

3. Consider the following three data points: $(-\pi, 0)$, $(0, 0)$, $(\pi/2, 1)$.

(a) **(10 points)** Write the Lagrange form of the quadratic polynomial that interpolates the given data.

Solution: The Lagrange form of the interpolation polynomial through three points $x_i, f(x_I)$, $i = 0, 1, 2$, is written as:

$$Q_2(x) = f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)l_2(x).$$

Here, the only nonzero value is $f(x_2) = 1$. Hence,

$$Q_2(x) = l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x + \pi)x}{(\pi/2 + \pi)(\pi/2)} = \frac{4x(x + \pi)}{3\pi^2}.$$

(b) **(10 points)** Repeat part (a) with Newton's form. Compute all the divided differences.

Solution: Newton's form of the interpolation polynomial is:

$$Q_2(x) = f(-\pi) + f[-\pi, 0](x + \pi) + f[-\pi, 0, \pi/2](x + \pi)x.$$

Here, $f(-\pi) = 0$ and

$$f[-\pi, 0] = \frac{f(0) - f(-\pi)}{0 - (-\pi)} = 0.$$

Also

$$f[0, \pi/2] = \frac{f(\pi/2) - f(0)}{\pi/2 - 0} = \frac{1 - 0}{\pi/2} = \frac{2}{\pi}.$$

Finally,

$$f[-\pi, 0, \pi/2] = \frac{f[0, \pi/2] - f[-\pi/0]}{\pi/2 - (-\pi)} = \frac{\frac{2}{\pi}}{3\pi/2} = \frac{4}{3\pi^2}.$$

Hence,

$$Q_2(x) = \frac{4x(x + \pi)}{3\pi^2},$$

which is identical to the result of part (a). As should be.

(c) **(5 points)** Assuming that the given data points were sampled from $f(x) = \sin(x)$, find an expression for the interpolation error.

Solution:

The interpolation error is given by

$$f(x) - Q_2(x) = \frac{f^3(\xi)}{3!}(x - x_0)(x - x_1)(x - x_2),$$

where ξ is an intermediate point in the interval. In our case: $-\pi < \xi < \pi/2$. For $f(x) = \sin(x)$, $f^3(x) = -\cos(x)$, which means that the interpolation error can be written as:

$$f(x) - Q_2(x) = \frac{-\cos(\xi)}{3!}(x + \pi)x(x - \pi/2).$$