AMSC/CMSC 460: Final Exam

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 2 hours.
- Good luck!

Additional instructions:

- You should solve only 6 out of the 7 problems. Each problem = 10 points.
- No extra credit will be given for solving more than 6 problems.
- If you solve more than 6 problems, you should clearly indicate which problems you would like to be graded - otherwise, the first 6 problems in each part will be graded.

Solve 6 problems out of the following 7 problems

1. Find the most accurate approximation to the second derivative, $f''(x)$, using $f(x-2h), f(x), f(x+4h)$. What is the order of accuracy of this approximation?

   Solution:

   Using the method of undetermined coefficients, we let

$$f''(x) \approx Af(x-2h) + Bf(x) + Cf(x+4h).$$

   We now write the Taylor expansions:

$$f(x-2h) = f(x) - 2hf'(x) + \frac{(2h)^2}{2}f''(x) - \frac{(2h)^3}{6}f'''(\xi_1),$$

   and

$$f(x+4h) = f(x) + 4hf'(x) + \frac{(4h)^2}{2}f''(x) + \frac{(4h)^3}{6}f'''(\xi_2).$$

   This leads to the following system:

$$\begin{cases} A + B + C = 0, \\ -2A + 4C = 0, \\ \frac{h^2}{2}(4A + 16C) = 1. \end{cases}$$

   The solution of this system is $A = \frac{1}{6h^2}$, $B = -\frac{1}{4h^2}$, $C = \frac{1}{12h^2}$, which means that the approximation is

$$f''(x) \approx \frac{2f(x-2h) - 3f(x) + f(x+4h)}{12h^2} + O(h).$$

2. Let $D(h)$ be a first-order approximation to $f'(x)$ such that

$$f'(x) = D(h) + C_1 h + C_2 h^2 + \ldots$$

   (a) Use Richardson's extrapolation to find a second-order approximation of $f'(x)$.

   Solution:

   By changing h to $2h$ we have

$$f'(x) = D(2h) + C_1 2h + C_2 (2h)^2 + \ldots$$

   Hence

$$f'(x) = 2D(h) - D(2h) + O(h^2).$$

(b) What is the result of part (a) if

$$D(h) = \frac{f(x+h) - f(x-3h)}{4h}.$$

Solution:

$$2D(h) - D(2h) = 2\frac{f(x+h) - f(x-3h)}{4h} - \frac{f(x+2h) - f(x-6h)}{8h}$$

3. Find a linear polynomial, $P_1^*(x)$, that minimizes

$$\int_{-1}^{1} \frac{(x^2 - Q_1(x))^2}{\sqrt{1-x^2}} dx,$$

among all polynomials $Q_1(x)$ of degree ≤ 1.

Solution:

This is a weighted least squares problem. The orthogonal polynomials are $T_0 = 1$ and $T_1 = x$. The solution of this problem is given by

$$P_1^*(x) = c_0 T_0(x) + c_1 T_1(x),$$

with

$$c_0 = \frac{\langle x^2, T_0 \rangle_w}{\|T_0\|_w^2} = \frac{\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} dx}{\pi} = \frac{\frac{\pi}{2}}{\pi} = \frac{1}{2},$$

and

$$c_1 = \frac{\langle x^2, T_1 \rangle_w}{\|T_1\|_w^2} = \frac{\int_{-1}^{1} \frac{x^2 \cdot x}{\sqrt{1-x^2}} dx}{\frac{\pi}{2}} = 0.$$

Hence $P_1^*(x) = \frac{1}{2}$.

4. (a) Find a quadrature of the form

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} dx = A_0 f(x_0) + A_1 f(x_1),$$

that is exact for all polynomials of degree ≤ 3.

Solution: The orthogonal polynomials that correspond to this problem are the Hermite polynomials: $H_0 = 1$, $H_1 = 2x$, and $H_2 = 2xH_1 - 2H_0 = 4x^2 - 2$. Since we need to quadrature points, we are looking for the roots of $H_2(x)$, which are $\pm\frac{1}{\sqrt{2}}$. All that remains is to find the coefficients A_0 and A_1, which we do by setting the quadrature to be exact for 1 and x:

$$\sqrt{\pi} = \int_{-\infty}^{\infty} 1 e^{-x^2} dx = A_0 + A_1,$$

and

$$0 = \int_{-\infty}^{\infty} xe^{-x^2} dx = A_0(-\frac{1}{\sqrt{2}}) + A_1\frac{1}{\sqrt{2}}.$$

Hence $A_0 = A_1 = \frac{\sqrt{\pi}}{2}$, and the approximation is

$$\int_{-\infty}^{\infty} f(x)e^{-x^2} dx \approx \frac{\sqrt{\pi}}{2}\left[f\left(-\frac{1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{2}}\right)\right].$$

(b) Use the result of part (a) to approximate $\int_{-\infty}^{\infty} x^6 e^{-x^2} dx$.

Solution:

$$\frac{\sqrt{\pi}}{2}\left[\left(-\frac{1}{\sqrt{2}}\right)^6 + \left(\frac{1}{\sqrt{2}}\right)^6\right] = \frac{\sqrt{\pi}}{8}.$$

5. Consider the ODE $y'(t) = f(t, y(t))$ together with the initial condition $y(0) = y_0$.

   (a) Write an equivalent integral formulation to the ODE and show how to obtain Euler's method using a rectangular quadrature.

   Solution:
   Integrating the ODE from t to $t+h$ we get

   $$y(t+h) = y(t) + \int_t^{t+h} f(s, y(s))ds.$$

   Replacing the integral with a rectangular quadrature, we get Euler's method: $w_0 = y_0$, and $w_{i+1} = w_i + hf(t_i, w_i)$.

   (b) Let $f(t, y(t)) = t^2 y(t)$. Compute two iterations of Euler's method for the ODE $y' = f(t, y(t))$, starting from $y(0) = 1$. Assume that the time step is $h = 0.1$.

   Solution: $w_0 = 1$. Hence $w_1 = w_0 + hf(0, 1) = w_0 = 1$, and $w_2 = w_1 + 0.1f(0.1, 1) = 1 + (0.1)^3$.

6. Let $f(x) = x^4$ in $[-1, 1]$.

   (a) Write the Lagrange form of the interpolating polynomial $P_2(x)$, of degree ≤ 2, that interpolates the values of $f(x)$ at 3 Chebyshev points.

   Solution:
   The problem requires 3 Chebyshev points which means that we need to find the roots of the cubic Chebyshev polynomial, $T_3(x)$. Since $T_0 = 1$ and $T_1 = x$, $T_2 = 2x^{-}1$, and $T_3 = 4x^3 - 3x$. Hence the roots we are looking for are $\pm\frac{\sqrt{3}}{2}, 0$. The values of the given function at these points are: $f(0) = 0$,

$f(-\sqrt{3}/2) = f(\sqrt{3}/2) = 9/16$. Hence, the Lagrange form of the interpolating polynomial is:

$$P_2(x) = \frac{9}{16}\frac{(x-0)(x-\frac{\sqrt{3}}{2})}{(-\frac{\sqrt{3}}{2}-0)(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2})} + \frac{9}{16}\frac{(x-0)(x+\frac{\sqrt{3}}{2})}{(\frac{\sqrt{3}}{2}-0)(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2})} = \frac{3}{4}x^2.$$

(b) Explain the advantages of interpolating at Chebyshev points.

Solution:
The interpolation error is given by

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi_n)}{(n+1)!}\prod_{i=0}^{n}(x-x_i).$$

Interpolating in Chebyshev points minimizes the product term in the error.

7. Let $f(x) = e^{-x} - x$.

(a) Prove that $f(x)$ must have an least one root in the interval $[0, 10]$.

Solution: $f(x)$ is a continuous function. $f(0) > 0$, and $f(10) < 0$. According to the midvalue theorem, there is at least one point $c \in (0, 10$ for which $f(c) = 0$.

(b) Explain why $f(x)$ has only one root in the interval $[0, 10]$. Any root of $f(x)$ satisfies $e^{-x} = x$. Plot e^{-x} and x and see that they intersect exactly once.

Solution:

(c) Write Newton's method for approximating a root of $f(x)$, and compute two iterations of the method, starting from $x_0 = 1$.

Solution:
Newton's method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n}).$$

With $x_0 = 1$, we have

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{e^{-1}-1}{-e^{-1}-1} = \frac{2}{1+e}.$$

And

$$x_2 = \frac{2}{1+e} - \frac{e^{-\frac{2}{1+e}} - \frac{2}{1+e}}{-e^{-\frac{2}{1+e}} - 1}.$$

- Chebyshev polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) = 0, \ \forall n \geq 1.$$

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = 0, \quad m \neq n.$$

$$\int_{-1}^{1} \frac{(T_n(x))^2}{\sqrt{1-x^2}} dx = \begin{cases} \pi, & n = 0, \\ \frac{\pi}{2}, & n = 1, 2, \ldots \end{cases}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C, \qquad \int \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2}(\arcsin x - x\sqrt{1-x^2}) + C.$$

- Hermite polynomials

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \ \forall n \geq 1$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x) = \delta_{nm} 2^n n! \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = \Gamma\left(\frac{m+1}{2}\right), \quad \text{for even } m$$

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(3/2) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma(5/2) = \tfrac{3}{4}\sqrt{\pi}.$$