AMSC/CMSC 460: Midterm 1 Solutions

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 75 minutes
- Good luck!

Problems: (Each problem = 9 points)

1. Let $f(x) = x^4 - 3x^2 + 2$.

   (a) Let $P_2(x)$ be a polynomial of degree ≤ 2 that interpolates $f(x)$ at $x_0 = -1, x_1 = 0, x_2 = 1$. Write $P_2(x)$ in Lagrange form.

   Solution: The values of $f(x)$ at the interpolation points are: $f(-1) = 0$, $f(0) = 2$, and $f(1) = 0$. Hence the Lagrange form of $P_2(x)$ is

   $$P_2(x) = 2\frac{(x+1)(x-1)}{(0+1)(0-1)} = 2(1-x^2).$$

   (b) Let $P_3(x)$ be a polynomial of degree ≤ 3 that interpolates $f(x)$ at $x_0 = -1, x_1 = 0, x_2 = 1, x_3 = 2$. Write $P_3(x)$ in Newton form.

   Solution: The fourth interpolation point adds the value $f(2) = 16-12+2 = 6$. To write the interpolant in Newton form we need to compute the divided differences. $f[-1] = f(-1) = 0$, $f[-1,0] = (f(0) - f(-1))/(0-(-1)) = 2$. $f[0,1] = (f(1) - f(0))/(1-0) = -2$, $f[1,2] = (f(2) - f(1))/(2-1) = 6$. Also $f[-1,0,1] = (f[0,1] - f[-1,0])/(1-(-1)) = -4/2 = -2$ and $f[0,1,2] = (f[1,2] - f[0,1])/(2-0) = 8/2 = 4$. Finally $f[-1,0,1,2] = (f[0,1,2] - f[-1,0,1])/(2-(-1)) = 6/3 = 2$. This means that the interpolation polynomial in Newton form is:

   $$\begin{aligned} P_3(x) &= f[-1] + f[-1,0](x+1) + f[-1,0,1](x+1)x \\ &\quad + f[-1,0,1,2](x+1)x(x-1) \\ &= 2(x+1) - 2(x+1)x + 2(x+1)x(x-1). \end{aligned}$$

   (c) Let $P_4(x)$ be a polynomial of degree ≤ 4 that interpolates $f(x)$ at $x_0 = -1, x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 30$. Without any calculations: what is $P_4(x)$? Justify your answer.

   Solution: Since $f(x)$ is a polynomial of degree 4, the uniqueness of the interpolating polynomials implies that $P_4(x)$ must be $f(x)$.

2. (a) Write the number 12.26 in base 2. (Compute the first 10 digits after the binary point).

   Solution:

   $$(12.26)_{10} = 1100.0100001010\ldots$$

(b) Explain how 12.26 can be represented as a floating point number on a 32-bit computer.

Solution: Numbers in a scientific binary notation are written as

$$\pm S \times 2^E.$$

Here S is the mantissa and E is the (signed) exponent.

To represent such numbers on a 32-bit machine, one bit will be used for the sign. 8 bits will be used for the (signed) exponent. This can be stored in a 2's complement representation. The remaining 23 bits will be used for representing the mantissa. Since all numbers (other than zero) have a leading digit of 1 in the mantissa, we will only store the digits after the binary dot. Special strings will be devoted to zero, infinity, and NaN.

In our case, the number 26 is binary scientific notation is

$$1.1000100001010\ldots \times 2^3.$$

So the representation is:

- sign bit (one bit): 0
- exponent (8 bits representing 3): 00000011
- mantissa (23 bits): 1000100001010...

(c) If instead of 32 bits, the computer has 64-bit words, what would you rather increase - the number of bits representing the exponent or the number of bits representing the mantissa? Explain.

Solution: While few bits can be added to the exponent, you want to add most of the bits to the mantissa to increase the accuracy. Adding bits to the exponent will increase the range of the numbers that can be represented. The IEEE standard specifies increasing both, however more bits are added to the mantissa.

3. Let $f(x) = e^x - 3x^2 - 2$.

(a) Find an interval that is guaranteed to include a root of $f(x)$. Justify your answer.

Solution: The function $f(x)$ is continuous. $f(0) = -2 < 0$ and $f(10) > 0$. According to the intermediate-value theorem, there exists at least one point c, $0 < c < 10$, for which $f(c) = 0$.

(b) Write Newton's method for approximating roots of $f(x)$. Starting from $x_0 = 0$, compute the first two iterations, x_1 and x_2. Do not simplify the expression you get for x_2.

Solution: Newton's method for root finding is the following: start with an initial guess x_0. For any $n \geq 0$,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

In our case, $f(x_n) = e^{x_n} - 3x_n^2 - 2$, while $f'(x_n) = e^{x_n} - 6x_n$. Now, if $x_0 = 0$ then

$$x_1 = x_0 - \frac{e^{x_0} - 3x_0^2 - 2}{e^{x_0} - 6x_0} = 0 - \frac{1-2}{1} = 1.$$

Similarly

$$x_2 = x_1 - \frac{e^{x_1} - 3x_1^2 - 2}{e^{x_1} - 6x_1} = 1 - \frac{e-5}{e-6} = \frac{e-6-e+5}{e-6} = \frac{1}{6-e}.$$

(c) Find a function $g(x)$ for which a root of $f(x)$ is a fixed point of $g(x)$.

Solution: If x is a root of $f(x) = e^x - 3x^2 - 2$, then $x^2 = (e^x - 2)/3$. Then $x = \pm\sqrt{(e^x - 2)/3}$ (assuming that $e^x > 2$. We can now define $g(x) = \sqrt{(e^x - 2)/3}$. Any point for which $g(x) = x$ is also a root of $f(x)$. Alternatively, write $e^x = 3x^2 + 2$, and set $g(x) = \log(3x^2 + 2)$. Other solutions are possible.

4. (a) Let $A = \begin{pmatrix} 9 & 3 & 6 \\ 3 & 17 & 6 \\ 6 & 6 & 30 \end{pmatrix}$. Find a Cholesky decomposition for A.

Solution: Let

$$L = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 1 & 5 \end{pmatrix}.$$

Then $A = LL^t$.

(b) Use the decomposition from part (a), to solve $Ax = b$ with $b = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$.

Solution: Let $y = L^t x$. Then

$$Ax = LL^t x = Ly = b.$$

That is

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}.$$

We solve $Ly = b$ for y using forward substitution. First $y_1 = 1$, then $y_1 + 4y_2 = 5$ which means that $y_2 = 1$. Finally $2y_1 + y_2 + 5y_3 = 8$, which means that $y_3 = 1$.
All that is left is to solve $L^t x = y$ for x in this case:

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We solve with backward substitution: $5x_3 = 1$ which means that $x_3 = 1/5$. Next, $4x_2 + x_3 = 1$ which means that $x_2 = 1/5$. Finally, $3x_1 + x_2 + 2x_3 = 1$. Hence $x_1 = 2/15$, i.e., the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{15} \\ \frac{1}{15} \\ \frac{1}{15} \end{pmatrix}.$$

(c) Use Gaussian elimination with scaled row pivoting to solve $Bx = c$ with $B = \begin{pmatrix} 3 & 9 & 6 \\ 9 & 3 & 6 \\ 3 & 5 & 30 \end{pmatrix}$. and $c = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$.

Solution: The scales of the rows are $(9, 9, 30)$. The corresponding ratios for the first column are $3/9, 9/9, 3/30$. Hence we switch the first and second row. The operations $R_2 \to R_2 - \frac{1}{3}R_1$ and $R_3 \to R_3 - \frac{1}{3}R_1$, bring the matrix to

$$\begin{pmatrix} 9 & 3 & 6 \\ 0 & 8 & 4 \\ 0 & 4 & 28 \end{pmatrix},$$

with the solution vector $\begin{pmatrix} 6 \\ 4 \\ 4 \end{pmatrix}$. We check the scaling of the second and third row and conclude that there is no need to switch them. Hence the next operation is $R_3 \to R_3 - \frac{1}{2}R_2$, which brings the matrix to

$$\begin{pmatrix} 9 & 3 & 6 \\ 0 & 8 & 4 \\ 0 & 0 & 26 \end{pmatrix},$$

with the solution vector $\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$. At this point we can solve by backward substitution, and the solution is $\begin{pmatrix} \frac{6}{13} \\ \frac{6}{13} \\ \frac{1}{13} \end{pmatrix}$.