AMSC 466: Midterm 1 - SOLUTIONS

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September 28, 2010

Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 60 minutes
- Good luck!

Problems:

1. (10 points) Let $f(x) = x^4 - 1$.

   (a) Compute the Lagrange form of the interpolating polynomial $Q_2(x)$ of degree ≤ 2 that interpolates $f(x)$ at $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

   Solution: We start by computing the values of $f(x)$ at the given points: $f(x_0) = f(-1) = 0$, $f(x_1) = f(0) = -1$, $f(x_2) = f(1) = 0$. Hence, in the Lagrange form of the interpolating polynomial there is only one nonzero term: the term with $f(x_1)$. This polynomial is:

   $$Q_2(x) = f(x_1)\frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = (x + 1)(x - 1) = x^2 - 1.$$

   (b) Sketch the function $f(x)$ and the interpolant $Q_2(x)$.

   Solution: easy

   (c) Write the error term for this interpolation, i.e., the difference $f(x) - Q_2(x)$. Assuming that $x \in [-1, 1]$, find any bound on the error term that only depends on x.

   Solution: We know that the difference between $f(x)$ and $Q_2(x)$ is given by

   $$f(x) - Q_2(x) = \frac{1}{3!}f^{(3)}(\xi)(x - x_0)(x - x_1)(x - x_2),$$

   where ξ is in an interval that contains x, x_0, x_1, x_2. Since $f'''(x) = 24x$, and $x \in [-1, 1]$, we know that $|f'''(\xi)| \leq 24$. Therefore

   $$|f(x) - Q_2(x)| \leq 4|(x + 1)x(x - 1)|.$$

2. (10 points) Let

$$f(x) = x - \log_{10}(x+1) - 3$$

(a) Prove that x has at least one root in the interval $[0, 9]$.

Solution: $f(x)$ is a continuous function in the interval $[0, 9]$. $f(0) = -3 < 0$ and $f(9) = 9 - 1 - 4 = 5 > 0$. Hence according to the intermediate value theorem there must exist at least one point $x^* \in (0, 9)$ such that $f(x^*) = 0$.

(b) Compute $f'(x)$. Recall that $\frac{d}{dx}\log_{10}(x) = \frac{1}{x \ln 10}$.

Solution:

$$f'(x) = 1 - \frac{1}{(x+1)\ln 10}.$$

(c) Using part (b), prove that $f(x)$ has only **one** root in the interval $[0, 9]$.

Solution: Since in $[0, 9]$, $f'(x) > 0$, the function $f(x)$ is monotonically increasing. Since it is negative at $x = 0$ and positive at $x = 9$ and since it is continuous, it must have only one root in that interval.

(d) Write Newton's method for finding the roots of the given $f(x)$.

Solution: Starting from an initial point x_0,

$$x_{n+1} = x_n - \frac{x_n - \log_{10}(x_n+1) - 3}{1 - \frac{1}{(x_n+1)\ln 10}}, \qquad n \geq 0.$$

(e) Denote the unique root of $f(x)$ by x^* and consider an initial value $x_0 > x^*$. Starting from any such x_0, do you expect Newton's method to converge to the root of $f(x)$? Explain without proof. (Hint: what is $f''(x)$?)

Solution: The second derivative of $f(x)$ is

$$f''(x) = \frac{1}{(x+1)^2 \ln 10}.$$

Hence, $f''(x) > 0$ in this interval, so that $f(x)$ is convex. We already know that $f(x)$ has a root in the interval. The convexity of the function will imply that if we start with $x_0 > x^*$, Newton's methods will thus provide x_n that converge to the root x^*. This can be justified in different ways, but note that it is impossible to use the theorem about the convergence of Newton's method for convex and monotone functions as is, since that theorem is a global theorem, and in this case, the function $f(x)$ is not defined for $x < -1$.

3. (5 points)

What is the divided difference $f[1, 2, 3, 4, 5, 6, 7, 8]$, for the data

x	1	2	3	4	5	6	7	8
$f(x)$	1	2	3	4	5	6	7	8

The maximum score for this problem is 5 points if you can answer it without any computations. If you compute anything, the maximum score is 3.

Solution: The divided difference $f[1, 2, 3, 4, 5, 6, 7, 8]$ is the coefficient of x^7 the interpolating polynomial of degree ≤ 7 that interpolates the given data. Since the function x interpolates the data, due to the uniqueness of the polynomial of degree ≤ 7 that interpolates the given 8 points, this function (x) must be the interpolating polynomials. Hence the given divided difference must be 0.