AMSC 466: Midterm 1

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 75 minutes
- Good luck!

Problems: (Each problem = 10 points. Maximum total points = 40)

1. (a) Verify directly that for any three distinct points x_0, x_1, and x_2,

$$f[x_0, x_1, x_2] = f[x_2, x_0, x_1].$$

Solution:

$$\begin{aligned}
f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0} = \\
&= \frac{(x_1 - x_0)(f(x_2) - f(x_1)) - (x_2 - x_1)(f(x_1) - f(x_0))}{(x_2 - x_1)(x_1 - x_0)(x_2 - x_0)} = \\
&= \frac{(x_1 - x_0)(f(x_2)) + (x_0 - x_2)f(x_1) + (x_2 - x_1)f(x_0)}{(x_2 - x_1)(x_1 - x_0)(x_2 - x_0)}.
\end{aligned}$$

while

$$\begin{aligned}
f[x_2, x_0, x_1] &= \frac{f[x_0, x_1] - f[x_2, x_0]}{x_1 - x_2} = \frac{\frac{f(x_1)-f(x_0)}{x_1-x_0} - \frac{f(x_0)-f(x_2)}{x_0-x_2}}{x_1 - x_2} = \\
&= \frac{(x_0 - x_2)(f(x_1) - f(x_0)) - (x_1 - x_0)(f(x_0) - f(x_2))}{(x_1 - x_0)(x_0 - x_2)(x_1 - x_2)} = \\
&= \frac{(x_1 - x_0)(f(x_2)) + (x_0 - x_2)f(x_1) + (x_2 - x_1)f(x_0)}{(x_2 - x_1)(x_1 - x_0)(x_2 - x_0)},
\end{aligned}$$

and we conclude that $f[x_0, x_1, x_2] = f[x_2, x_0, x_1]$.

(b) Assume that x_0, x_1, x_2 are equally spaced, i.e., $x_1 = x_0 + h$ and $x_2 = x_1 + h$, with $h > 0$. Compute $f[x_0, x_1, x_2]$.

Solution:

$$\begin{aligned}
f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0} = \\
&= \frac{1}{2}\frac{f(x_2) - 2f(x_1) + f(x_0)}{h^2}.
\end{aligned}$$

2. Consider two distinct points $x_0 < x_1$ and a twice continuously differentiable function $f(x)$. Let $Q_1(x)$ be the linear interpolant through $f(x_0)$ and $f(x_1)$.

   (a) Write the Lagrange form of $Q_1(x)$.

   Solution:

$$Q_1(x) = f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0}.$$

   (b) Using the general interpolation error formula, show that the interpolation error $|f(x) - Q_1(x)|$ in the interval $[x_0, x_1]$ is bounded by $\frac{1}{8}h^2 M$, where $h = x_1 - x_0$ and $M = \max_{x_0 \le x \le x_1} |f''(x)|$.

   Solution:

   The polynomial interpolation error in this case is given by

$$f(x) - Q_1(x) = \frac{1}{2!} f''(\xi)(x - x_0)(x - x_1),$$

   with ξ being an intermediate point in the interval $\xi \in (x_0, x_1)$. Hence

$$|f(x) - Q_1(x)| \le \frac{1}{2} M \max_{x_0 \le x \le x_1} |(x - x_0)(x - x_1)|,$$

   with $M = \max_{x_0 \le x \le x_1} |f''(x)|$. If we let $g(x) = (x - x_0)(x - x_1)$, then $g'(x) = 2x - (x_0 + x_1)$, and $g''(x) = 2 \ge 2$. Hence $g(x)$ obtains a maximum at $x = (x_0 + x_1)/2$. This means that

$$\max_{x_0 \le x \le x_1} |(x - x_0)(x - x_1)| = \left| \left(\frac{x_0 + x_1}{2} - x_0 \right) \left(\frac{x_0 + x_1}{2} - x_1 \right) \right| = \frac{h}{2} \cdot \frac{h}{2}.$$

   Therefore

$$|f(x) - Q_1(x)| \le \frac{1}{2} M \frac{h^2}{4} = \frac{1}{8} M h^2.$$

3. (a) Show that Newton's method for finding $\sqrt{R}$ can be written as

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right).$$

Solution:

To compute $\sqrt{R}$ let $f(x) = x^2 - R$. Then the roots of $f(x)$ are $\pm\sqrt{R}$. Newton's method for this $f(x)$ is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - R}{2x_n} = \frac{2x_n^2 - x_n^2 + R}{2x_n} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right).$$

(b) Let $R = AB$. Using two iterations of the formula from part (a) starting with $x_0 = A$, show that an approximation of $\sqrt{AB}$ can be written as

$$\sqrt{AB} \approx \frac{A+B}{4} + \frac{AB}{A+B}.$$

What happens if $x_0 = \frac{A+B}{2}$?

Solution:

We use the result from part (a) with $R = AB$ and $x_0 = A$. In this case

$$x_1 = \frac{1}{2}\left(A + \frac{R}{A}\right) = \frac{A+B}{2}.$$

This means that

$$x_2 = \frac{1}{2}\left(\frac{1}{2}\frac{A+B}{2} + \frac{R}{\frac{A+B}{2}}\right) = \frac{A+B}{4} + \frac{AB}{A+B},$$

which is the desired approximation of $\sqrt{AB}$. If $x_0 = \frac{A+B}{2}$, we reach the same approximation after one iteration. If we have two iterations starting from that point, we will get a better approximation of $\sqrt{AB}$ which can be computed.

4. Denote the successive intervals that arise in the bisection method by $[a_0, b_0]$, $[a_1, b_1]$, etc.

(a) Show that $a_0 \le a_1 \le a_2 \le \cdots$ and that $b_0 \ge b_1 \ge b_2 \ge \cdots$

Solution: This is the bisection method, see notes.

(b) Show that $b_n - a_n = 2^{-n}(b_0 - a_0)$.

Solution: A standard result for the bisection method, see notes.

(c) Show that for all n, $a_n b_n + a_{n-1} b_{n-1} = a_{n-1} b_n + a_n b_{n-1}$.

Solution:
In the bisection method either $a_{n-1} = a_n$ or $b_{n-1} = b_n$, and the result holds.