AMSC 466: Midterm 2 – SOLUTIONS

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 60 minutes
- Good luck!

Problems:

1. (10 points)

(a) Write down the conditions that should be satisfied so that the following function is a natural cubic spline on the interval $[0, 2]$:

$$s(x) = \begin{cases} f_1(x), & x \in [0,1], \\ f_2(x), & x \in [1,2]. \end{cases}$$

Solution: For $s(x)$ to be a cubic spline, $f_1(x)$ and $f_2(x)$ should be cubic polynomials that satisfy the following conditions: $f_1(1) = f_2(1)$, $f_1'(1) = f_2'(1)$, and $f_1''(1) = f_2''(1)$. In addition, in order for $s(x)$ to be a natural spline on $[0, 2]$ we must require that $f_1''(0) = 0$, and that $f_2''(2) = 0$.

(b) Determine the values of the coefficients a, b, c, d, and e so that the following $s(x)$ is a natural cubic spline on $[0, 2]$:

$$s(x) = \begin{cases} 1 + x - ax^2 + bx^3, & x \in [0,1], \\ c + d(x-1) + e(x-2)^2 + (x-2)^3, & x \in [1,2]. \end{cases}$$

Solution: We denote the function in $[0, 1]$ by $f_1(x)$ and the function in $[1, 2]$ by $f_2(x)$. We then have

$$\begin{aligned} f_1(x) &= 1 + x - ax^2 + bx^3, \\ f_1'(x) &= 1 - 2ax + 3bx^2, \\ f_1''(x) &= -2a + 6bx, \\ f_2(x) &= c + d(x-1) + e(x-2)^2 + (x-2)^3, \\ f_2'(x) &= d + 2e(x-2) + 3(x-2)^2, \\ f_2''(x) &= 2e + 6(x-2). \end{aligned}$$

For $s(x)$ to be a natural spline we ask that $f_1''(0) = 0$. This implies that $a = 0$. Also $f_2''(2) = 0$, which implies that $e = 0$.
The continuity of $s(x)$ at $x = 1$ implies that:

$$2 + b = c - 1.$$

The continuity of $s'(x)$ at $x = 1$ implies that:

$$1 + 3b = d + 3.$$

Finally, the continuity of $s''(x)$ at $x = 1$ implies that

$$6b = -6.$$

These equations are satisfies when $b = -1$, $c = 2$, and $d = -5$.

2. (10 points)

(a) Write the Hermite interpolation polynomial to $f(x)$ based on the given values of $f(a), f'(a), f(b)$.

Solution: The Newton form of the Hermite interpolation polynomial in this case is:

$$P_2(x) = f(a) + f[a,a](x-a) + f[a,a,b](x-a)^2.$$

The divided differences are:

$$f[a,a] = f'(a),$$

and

$$f[a,a,b] = \frac{f[a,b]-f[a,a]}{b-a} = \frac{\frac{f(b)-f(a)}{b-a} - f'(a)}{b-a} = \frac{f(b)-f(a)-(b-a)f'(a)}{(b-a)^2.}$$

Hence, the requested interpolant is:

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f(b)-f(a)-(b-a)f'(a)}{(b-a)^2}(x-a)^2.$$

(b) Based on the result of (a), write an approximation of

$$\int_a^b f(x)dx.$$

Solution: The idea is to approximate the integral of $f(x)$ on the interval $[a,b]$ by integrating the Hermite interpolant $P_2(x)$ on this interval. In order to do that, we note that

$$\int_a^b (x-a)dx = \frac{1}{2}(b-a)^2,$$

and

$$\int_a^b (x-a)^2 dx = \frac{1}{3}(b-a)^3.$$

Hence

$$\begin{aligned}\int_a^b f(x)dx &\approx \int_a^b P_2(x)dx \\ &= \int_a^b \left[f(a) + f'(a)(x-a) + \frac{f(b)-f(a)-(b-a)f'(a)}{(b-a)^2}(x-a)^2 \right] dx \\ &= f(a)(b-a) + f'(a)\frac{1}{2}(b-a)^2 + \frac{f(b)-f(a)-(b-a)f'(a)}{(b-a)^2}\frac{1}{3}(b-a)^3,\end{aligned}$$

which is the desired approximation (that can be further simplified).

3. (a) (5 points) Find the first two orthogonal polynomials $p_0(x)$ and $p_1(x)$, with respect to the weight function $w(x) \equiv 1$ on $[2, 5]$. Show your calculations.

Solution: We will use the Gram-Schmidt orthogonalization process to replace the polynomials $\{1, x, x^2\}$ with polynomials that are orthogonal with respect to the inner product

$$(f, g) = \int_2^5 f(x)g(x)dx.$$

We now set the first polynomial as

$$P_0(x) = 1.$$

The next polynomial is of the form

$$P_1(x) = x - c_0 P_0(x) = x - c_0.$$

To find the constant c_0, we invoke the orthogonality condition, i.e., we require that

$$\int_2^5 P_0(x)P_1(x)dx = 0.$$

We thus have

$$0 = \int_2^5 1 \cdot (x - c_0)dx = \int_2^5 xdx - c_0 \int_2^5 dx = \frac{1}{2}(5^2 - 2^2) - 3c_0 = \frac{21}{2} - 3c_0,$$

which means that $c_0 = 7/2$, and the polynomial $P_1(x)$ is

$$P_1(x) = x - \frac{7}{2}.$$

(b) (5 points) Find the polynomial $Q_0(x)$ of degree zero that minimizes

$$\int_2^5 [e^{-x} - Q_0(x)]^2 dx.$$

Solution: This is a very simple weighted least squares problem. The polynomial that we are looking for is $Q_0 = a_0$. To determine the value of the constant a_0 we compute

$$a_0 = \frac{\int_2^5 f(x)P_0(x)dx}{\int_2^5 (P_0)^2 dx}.$$

Here $P_0 = 1$ and $f(x) = e^{-x}$. So the integrals are:

$$\int_2^5 e^{-x}dx = -e^{-x}\Big|_2^5 = e^{-2} - e^{-5},$$

and

$$\int_2^5 dx = 3$$

so the polynomial $Q_0(x)$ is

$$Q_0(x) = \frac{e^{-2} - e^{-5}}{3}.$$